It is evident that for any finite set $\{c_n\}$ of points in the complex plane, we can associate a polynomial $p(z) = \prod_n (z - c_n)$ whose zeros are precisely those points in the set. Conversely, as a consequence of the Fundamental Theorem of Algebra, any polynomial function $p(z)$ in the complex plane can be factored as $p(z) = a \prod_n (z - c_n)$ where a is a non-zero constant and $\{c_n\}$ is the set of zeros of $p(z)$.

Weierstrass Factorization Theorem studies what happens when the set is not finite. It provides representations of entire functions, as products involving their zeros.

Remark 1. *Inspiration for this lesson comes mainly from [1].*

We start with some simpler cases:

Theorem 1. *If an entire function $f(z)$ has no zeros, then $f(z) = e^{g(z)}$ for some entire function $g(z)$.*

Proof. Define the function $h(z)$ as the logarithmic derivative of $f(z)$:

$$h(z) = \frac{f'(z)}{f(z)} = \frac{\mathrm{d}}{\mathrm{d}z} \log f(z)$$

$h(z)$ is entire, as $f(z)$ has no zeros, therefore:

$$\int_0^z h(s)\mathrm{d}s = \log f(s)\big|_0^z = \log f(z) - \log f(0)$$

$$\Downarrow$$

$$\log f(z) = \int_0^z h(s)\mathrm{d}s + \log f(0)$$

$$\Downarrow$$

$$f(z) = e^{\int_0^z h(s)\mathrm{d}s + \log f(0)} = e^{g(z)}.$$

□

Theorem 2. *If an entire function $f(z)$ has a finite number of zeros, say $z_1, \cdots, z_m$, with k_j, $j = 1, \cdots m$ the order of z_j. Then $f(z)$ is of the form:*

$$f(z) = (z - z_1)^{k_1} \cdots (z - z_m)^{k_m} e^{g(z)}$$

where $g(z)$ is an entire function.

Proof. Simply note that the function:

$$F(z) = \frac{f(z)}{(z - z_1)^{k_1} \cdots (z - z_m)^{k_m}}$$

is an entire function with no zeros. Hence Theorem 1 implies $F(z) = e^{g(z)}$ and the result follows. □

Remark 2. *It is evident that Theorem 1 is a special case of Theorem 2.*

Let's tackle now the case of entire functions with an infinite number of zeros, say $z_1, z_2, \cdots, z_n, \cdots$.
To do this, we need to define and study **Weierstrass's elementary Functions**:

$$E_0(z) = (1 - z), \qquad E_l(z) = (1 - z) e^{z + \frac{z^2}{2} + \cdots + \frac{z^l}{l}}, \qquad l = 1, 2, \cdots$$

Theorem 3. *Each elementary function $E_l(z)$, $l = 0, 1, 2, \cdots$ is an entire function with a simple zero at $z = 1$. Also:*

1. *$E_l'(z) = -z^l e^{z + \frac{z^2}{2} + \cdots + \frac{z^l}{l}}$.*
2. *If $E_l(z) = \sum_{j=0}^{\infty} a_j z^z$ is the power series expansion of $E_l(z)$ about $z = 0$, then $a_0 = 1$, $a_1 = a_2 = \cdots = a_l = 0$ and $a_j \le 0$ for $j > l$.*
3. *If $|z| \le 1$, then $|E_l(z) - 1| \le |z|^{l+1}$.*

Proof. 1.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z} E_l(z) &= -e^{z + \frac{z^2}{2} + \cdots + \frac{z^l}{l}} + e^{z + \frac{z^2}{2} + \cdots + \frac{z^l}{l}} (1 + z + z^2 + \cdots + z^{l-1})(1 - z) \\ &= -z^l e^{z + \frac{z^2}{2} + \cdots + \frac{z^l}{l}}. \end{aligned} \tag{1}$$

2. The coefficients of the power series expansion of an analytic function f about $z = c$ are:

$$a_n = \frac{f(c)}{n!}.$$

Therefore $a_0 = E(0) = 1$. On the other hand, $E_l'(z)$ has a zero of multiplicity l at 0, hence, since term-by-term differentiation is permissible, it follows that:

$$a_j = \frac{E_l^j(0)}{j!} = 0, \qquad \text{for } j = 1, 2, \cdots, l$$

where $E_l^j(z)$ is the $j-$th derivative of $E_l(z)$.

For $j > l$, we can see that in the expansion of $E_l'(z) = -z^l e^{z + \frac{z^2}{2} + \cdots + \frac{z^l}{l}}$ the coefficient of each z^j is a nonpositive real number.

3. Using part 2. we know that:

$$|E_l(z)-1| = \left|\sum_{j=0}^{\infty} a_j z^j - 1\right| = \left|\sum_{j=l+1}^{\infty} a_j\right| \le \sum_{j=l+1}^{\infty} |a_j||z|^j \le |z|^{l+1} \sum_{j=l+1}^{\infty} (-a_j)|z|^{j-l-1}.$$

Notice now that $E_l(1) = 0 = 1 + \sum_{j=l+1}^{\infty} a_j \Rightarrow -\sum_{j=l+1}^{\infty} a_j = 1$. Hence, for $|z| \le 1$:

$$|E_l(z) - 1| \le |z|^{l+1} \sum_{j=l+1}^{\infty} (-a_j)|z|^{j-l-1} \le |z|^{l+1} \sum_{j=l+1}^{\infty} (-a_j) = |z|^{l+1}.$$

□

Lemma 1. *Let $\{z_n\}$ be a sequence of complex numbers such that: $z_n \neq 0$, $n = 1, 2, \cdots$ and $|z_n| \to \infty$ as $n \to \infty$.*

Then, there exists a sequence of nonnegative integers $\{l_n\}$ such that the series $\sum_{j=1}^{\infty} \left|\frac{z}{z_j}\right|^{l_j+1}$ is uniformly convergent on every closed disk $\overline{B}(0, r)$, $r < \infty$.

Proof. Fix $l_n = n - 1$ for $n = 1, 2, \cdots$. Since $|z_n| \to \infty$ as $n \to \infty$, there exists N large enough so that $|z_n| \ge 2r$ for $n \ge N$. For all these n and for $|z| \le r$ (i.e. $z \in \overline{B}(0, r)$), we have:

$$\left|\frac{z}{z_n}\right|^{l_n+1} = \left|\frac{z}{z_n}\right|^n \le \left(\frac{r}{2r}\right)^n = \frac{1}{2^n}.$$

This proves that we can use **Weierstrass's M-Test** to conclude that the series is uniformly convergent in $\overline{B}(0, r)$. □

We can finally state the main Theorem:

Theorem 4. ***Weierstrass's Factorization Theorem***
Let $\{z_n\}$ be a sequence of complex numbers such that $z_n \neq 0$ for $n = 1, 2, \cdots$ and $|z_n| \to \infty$ as $n \to \infty$. Let m be a nonnegative integer. Furthermore, let $\{l_n\}$ be a sequence of nonnegative integers such that the series $\sum_{j=1}^{\infty} \left|\frac{z}{z_j}\right|^{l_j+1}$ converges uniformly on compact subsets of the complex plane. (Notice that Lemma 1 implies that such $\{l_n\}$ can always be found). Then:

$$f(z) = z^m \prod_{j=1}^{\infty} E_{l_j}\left(\frac{z}{z_j}\right) \tag{2}$$

is an entire function. This function has zeros at $z = 0$ of multiplicity m and at each z_j, $j = 1, 2, \cdots$ of multiplicity k_j. Here k_j is the number of times z_j occurs in the sequence $\{z_n\}$.

Remark 3. *This Theorem is often stated in a different manner, we discuss it after the proof.*

Proof. Write $E_{l_n}\left(\frac{z}{z_n}\right)$ as:

$$E_{l_n}\left(\frac{z}{z_n}\right) = 1 + \left(E_{l_n}\left(\frac{z}{z_n}\right) - 1\right)$$

then the product in 2 converges uniformly and absolutely provided that the series $\sum_{j=1}^{\infty}\left(E_{l_n}\left(\frac{z}{z_n}\right) - 1\right)$ converges uniformly and absolutely on every disk of finite radius.

Remark 4. *This is a classic result in complex analysis, it can be found as Theorem 42.6 of [1].*

Let $|z| \leq r$. Then, given that $|z_n| \to \infty$, there exists N large enough so that $\frac{z}{z_n} \leq 1$ for $n \geq N$. Therefore, using point 3. of Theorem 3, we have

$$\left|E_{l_n}\left(\frac{z}{z_n}\right) - 1\right| \leq \left|\frac{z}{z_n}\right|^{l_n+1}.$$

Since we can choose l_n as in lemma 1, it follows that the series $\sum_{j=1}^{\infty}\left|\frac{z}{z_j}\right|^{l_n+1}$ converges uniformly on the closed disk $\overline{B}(0,r)$, hence the series $\left|E_{l_n}\left(\frac{z}{z_n}\right) - 1\right|$ converges uniformly and absolutely on $\overline{B}(0,r)$.
In conclusion, since $r > 0$ is arbitrary, we have proved that the series $\sum_{j=1}^{\infty}\left(E_{l_n}\left(\frac{z}{z_n}\right) - 1\right)$ converges uniformly and absolutely on every disk of finite radius, therefore the product in equation 2 converges and the limit function $f(z)$ is entire and has the prescribed zeros. □

Remark 5. *Notice that the sequence of positive integers $\{l_n\}$ is not necessarily unique, hence, so is the representation 2.*

Theorem 5. *Let $f(z)$ be an entire function with an infinite number of roots, let $\{\rho_n\}$ be the non-zero roots of $f(z)$ repeated with multiplicity. Let g be the order of the zero at $z = 0$. Then there exists a sequence of integers $\{l_n\}$ such that:*

$$f(z) = z^g \prod_{j=1}^{\infty} E_{l_j}\left(\frac{z}{\rho_j}\right). \tag{3}$$

Proof. To prove this factorization we simply need to remember that the sequence of non-zero roots $\{\rho_n\}$, ordered by absolute value satisfies $|\rho_n| \to 0$ as $n \to \infty$.

Remark 6. *This is a general property of analytic functions.*

Hence, we can choose $\{z_n\} = \{\rho_n\}$, $m = g$ and the sequence $\{l_n\}$ obtained with the lemma in Theorem 4. This yields exactly equation 3. □

POSITIVE INCREMENT

Thank you!

We hope this lesson has been beneficial in studying this interesting topic.
For more lessons or demonstrations, visit our website.

References

[1] Ravi P Agarwal, Kanishka Perera, and Sandra Pinelas. *An introduction to complex analysis*. Springer Science & Business Media, 2011.