Theorem 1. *The number of zeros of the Riemann Zeta Function $\zeta(s)$ in the critical strip, that have $0 < Im(s) < T$ is:*

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \mathcal{O}(\log(T)). \tag{1}$$

The proof that follows is a more detailed version of the one presented by W.Dittrichin in [1].

Proof. Assume that $T \geq 3$ and $\zeta(s) \neq 0$ for $Im(s) = T$.

Consider the rectangle R_T:

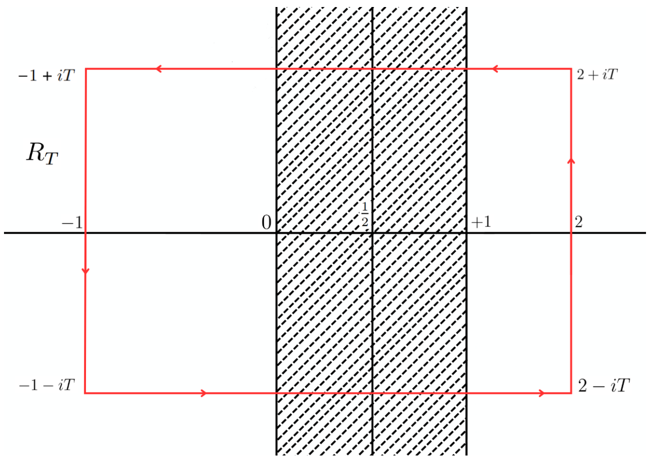

Figure 1: The rectangle R_T

We are going to proceed by using the argument principle on the entire function $\xi(s)$, remember that the Riemann ξ function is defined as:

$$\xi(s) := \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{2}$$

The reason we use this function is that its zeros are identical to the ones of the ζ function in the critical range.

Remark 1. *If the last statement isn't clear, we encourage you to explore our lesson on Riemann's original article for a deeper understanding.*

Therefore:

$$\frac{1}{2\pi i}\int_{\partial R_T}\frac{\xi'(s)}{\xi(s)}ds = Z$$

where Z is the number of zeros of $\xi(s)$ inside the contour ∂R_T and therefore the number of zeros of $\xi(s)$ in the critical strip.

Notice that $\xi(s)$ has real values for all $s\in\mathbb{R}$ and therefore satisfies

$$\xi(\overline{s}) = \overline{\xi(s)}$$

this implies that:

$$\xi(s) = 0 \iff \xi(\overline{s}) = 0.$$

By definition, $N(T)$ is only the number of zeros in the portion of the critical strip above the real line, hence we can write:

$$Z = 2N(T) = \frac{1}{2\pi i}\int_{\partial R_T}\frac{\xi'(s)}{\xi(s)}ds.$$

The functional equation $\xi(s) = \xi(1-s)$ implies that

$$-\frac{\xi'(1-s)}{\xi(1-s)} = \frac{\xi'(s)}{\xi(s)}$$

so, calling C_T' the left side of ∂R_T and C_T the right side (see Figure 2), we have:

$$\int_{C_T'}\frac{\xi'(s)}{\xi(s)}ds = \int_{C_T} -\frac{\xi'(1-s)}{\xi(1-s)}ds = \int_{C_T}\frac{\xi'(s)}{\xi(s)}ds$$

$$\Downarrow$$

$$2N(T) = \frac{1}{2\pi i}\int_{\partial R_T}\frac{\xi'(s)}{\xi(s)}ds = \frac{1}{2\pi i}\int_{C_T'}\frac{\xi'(s)}{\xi(s)}ds + \frac{1}{2\pi i}\int_{C_T}\frac{\xi'(s)}{\xi(s)}ds = 2\left[\frac{1}{2\pi i}\int_{C_T}\frac{\xi'(s)}{\xi(s)}ds\right].$$

Therefore, what we need to evaluate is:

$$N(T) = \frac{1}{2\pi i}\int_{C_T}\frac{\xi'(s)}{\xi(s)}ds. \tag{3}$$

Compute the logarithm of $\xi(s)$ using its definition (2):

$$\log(\xi(s)) = -\log(2) + \log(s) + \log(s-1) - \frac{s}{2}\log(\pi) + \log\Gamma\left(\frac{s}{2}\right) + \log\zeta(s)$$

differentiating both sides:

$$\frac{d}{ds}\log(\xi(s)) = \frac{\xi'(s)}{\xi(s)} = \frac{1}{s} + \frac{1}{s-1} - \frac{1}{2}\log(\pi) + \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + \frac{\zeta'(s)}{\zeta(s)}.$$

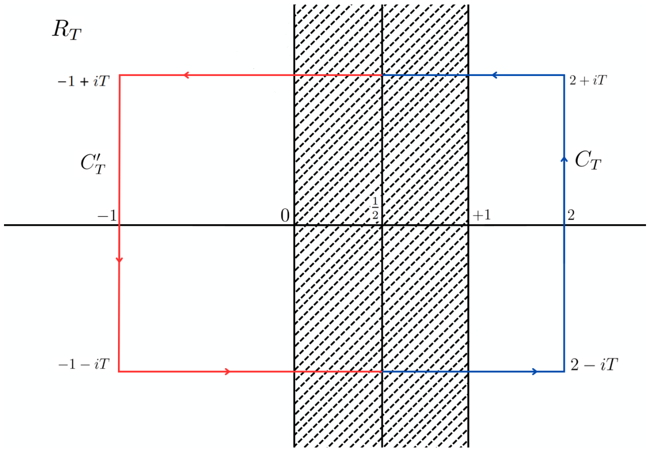

Figure 2: The rectangle R_T divided in C_T and $C_T^{'}$

Let's substitute this values in equation 3:

$$2\pi i N(T) = \int_{C_T} \left(\frac{1}{s} + \frac{1}{s-1}\right) ds - \frac{1}{2}\int_{C_T} \log(\pi) ds + \frac{1}{2}\int_{C_T} \frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} ds + \int_{C_T} \frac{\zeta'(s)}{\zeta(s)} ds. \tag{4}$$

Calculating the integrals separately we have:

$$\int_{C_T} \left(\frac{1}{s} + \frac{1}{s-1}\right) ds = \frac{1}{2}\int_{\partial R_T} \left(\frac{1}{s} + \frac{1}{s-1}\right) ds = \frac{1}{2} 2\pi i (1+1) = 2\pi i$$

where we used the Residue Theorem and the fact that $s = 0$ and $s = 1$ are two simple poles.

$$\frac{1}{2}\int_{C_T} \log(\pi) ds = \frac{1}{2}\log(\pi)\int_{C_T} ds = \frac{1}{2}\log(\pi)\left((2+iT) - (2-iT)\right) = iT\log(\pi).$$

While

$$\begin{aligned} \frac{1}{2}\int_{C_T} \frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} ds &= \frac{1}{2}\int_{C_T} \frac{d}{ds}\left[\log\Gamma\left(\frac{s}{2}\right)\right] = \left|\log\Gamma\left(\frac{s}{2}\right)\right|_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \\ &= \log\Gamma\left(\frac{1}{4} + i\frac{T}{2}\right) - \log\Gamma\left(\frac{1}{4} - i\frac{T}{2}\right) = \log\Gamma\left(\frac{1}{4} + i\frac{T}{2}\right) - \overline{\log\Gamma\left(\frac{1}{4} + i\frac{T}{2}\right)} \\ &= 2iIm\left(\log\Gamma\left(\frac{1}{4} + i\frac{T}{2}\right)\right) \end{aligned} \tag{5}$$

where we used the fact that $\Gamma(\overline{s}) = \overline{(\Gamma(s))} \Rightarrow \log\Gamma(\overline{s}) = \overline{\log\Gamma(s)}$.

Consider the expansion for $\log\Gamma(s)$ known as **Stirling's Series**:

$$\log\Gamma(s) = \left(s - \frac{1}{2}\right)\log(s) - s + \log(\sqrt{2\pi}) + \mathcal{O}\left(\frac{1}{s}\right)$$

to obtain:

$$
\begin{aligned}
&= 2iIm\left(\left(\frac{1}{4}+i\frac{T}{2}-\frac{1}{2}\right)\log\left(\frac{1}{4}+i\frac{T}{2}\right)-\left(\frac{1}{4}+i\frac{T}{2}\right)+\log(\sqrt{2\pi})+\mathcal{O}\left(\frac{1}{T}\right)\right)\\
&= 2iIm\left(\left(-\frac{1}{4}+i\frac{T}{2}\right)\log\left(i\frac{T}{2}\right)-i\frac{T}{2}+\log(\sqrt{2\pi})+\mathcal{O}\left(\frac{1}{T}\right)\right)\\
&= 2iIm\left(\left(-\frac{1}{4}+i\frac{T}{2}\right)\left[\log\left(\frac{T}{2}\right)+\log(i)\right]-i\frac{T}{2}+\log(\sqrt{2\pi})+\mathcal{O}\left(\frac{1}{T}\right)\right)\\
&= 2iIm\left(\left(-\frac{1}{4}+i\frac{T}{2}\right)\left[\log\left(\frac{T}{2}\right)+i\frac{\pi}{2}\right]-i\frac{T}{2}+\log(\sqrt{2\pi})+\mathcal{O}\left(\frac{1}{T}\right)\right)\\
&= 2iIm\left(-\frac{1}{4}\log\left(\frac{T}{2}\right)-i\frac{\pi}{8}+i\frac{T}{2}\log\left(\frac{T}{2}\right)-\frac{\pi T}{4}-i\frac{T}{2}+\log(\sqrt{2\pi})+\mathcal{O}\left(\frac{1}{T}\right)\right)\\
&= 2i\left(-\frac{\pi}{8}+\frac{T}{2}\log\left(\frac{T}{2}\right)-\frac{T}{2}+\mathcal{O}\left(\frac{1}{T}\right)\right)\\
&= 2\pi i\left(\frac{T}{2\pi}\log\left(\frac{T}{2}\right)-\frac{T}{2\pi}-\frac{1}{8}+\mathcal{O}\left(\frac{1}{T}\right)\right)
\end{aligned} \tag{6}
$$

we have arrived to an intermediate result:

$$2\pi iN(T) = 2\pi i - iT\log(\pi) + 2\pi i\left(\frac{T}{2\pi}\log\left(\frac{T}{2}\right)-\frac{T}{2\pi}-\frac{1}{8}+\mathcal{O}\left(\frac{1}{T}\right)\right)+\int_{C_T}\frac{\zeta'(s)}{\zeta(s)}ds$$

$$\Downarrow$$

$$N(T) = 1-\frac{T}{2\pi}\log(\pi)+\frac{T}{2\pi}\log\left(\frac{T}{2}\right)-\frac{T}{2\pi}-\frac{1}{8}+\mathcal{O}\left(\frac{1}{T}\right)+\frac{1}{2\pi i}\int_{C_T}\frac{\zeta'(s)}{\zeta(s)}ds. \tag{7}$$

Split the last term in two parts:

First:

$$\int_{2-iT}^{2+iT}\frac{\zeta'(s)}{\zeta(s)}ds = \int_{2-iT}^{2+iT}\frac{d}{ds}\log\zeta(s)ds = |\log\zeta(s)|_{2-iT}^{2+iT}.$$

Remember that the Riemann Zeta Function can be written as an Euler Product when $Re(s) > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_p\frac{1}{1-\frac{1}{p^s}} \tag{8}$$

$$\Downarrow$$

$$\log(\zeta(s)) = \log\left(\prod_p(1-p^{-s})^{-1}\right) = \sum_p\log\left((1-p^{-s})^{-1}\right) = -\sum_p\log\left((1-p^{-s})\right). \tag{9}$$

Using 9 and the fact that $\ln\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty}\frac{x^n}{n}$ we have:

$$|\log(\zeta(2+iT))| = \left|\sum_p\log\left(\frac{1}{1-p^{-2-iT}}\right)\right| \le \left|\sum_p\sum_{n=1}^{\infty}\frac{(p^{-2-iT})^n}{n}\right|$$

$$\leq \sum_p \sum_{n=1}^{\infty} \frac{(p^{-2})^n}{n} = \sum_p \log\left(\frac{1}{1-p^{-2}}\right) = \log(\zeta(2)) = \log\left(\frac{\pi^2}{6}\right).$$

Likewise:

$$|\log(\zeta(2-iT))| \leq \log\left(\frac{\pi^2}{6}\right).$$

Therefore:

$$\int_{2-iT}^{2+iT} \frac{\zeta'(s)}{\zeta(s)} ds = \mathcal{O}(1), \qquad \text{For } T \geq 3.$$

For the second part, using this time the fact that $\zeta(\bar{s}) = \overline{\zeta(s)}$ we have:

$$\int_{\frac{1}{2}-iT}^{2-iT} \frac{\zeta'(s)}{\zeta(s)} ds = \int_{\frac{1}{2}}^{2} \frac{\zeta'(\sigma - iT)}{\zeta(\sigma - iT)} d\sigma = \overline{\int_{\frac{1}{2}}^{2} \frac{\zeta'(\sigma + iT)}{\zeta(\sigma + iT)} d\sigma} = \overline{\int_{\frac{1}{2}+iT}^{2+iT} \frac{\zeta'(s)}{\zeta(s)} ds}$$

so that:

$$\frac{1}{2\pi i}\left(\int_{\frac{1}{2}-iT}^{2-it} \frac{\zeta'(s)}{\zeta(s)} ds + \int_{2+it}^{\frac{1}{2}+iT} \frac{\zeta'(s)}{\zeta(s)} ds\right) = \frac{1}{2\pi i}\left(\overline{\int_{\frac{1}{2}+iT}^{2+iT} \frac{\zeta'(s)}{\zeta(s)} ds} - \int_{\frac{1}{2}+iT}^{2+it} \frac{\zeta'(s)}{\zeta(s)} ds\right)$$
$$= -\frac{1}{\pi} Im\left(\int_{\frac{1}{2}+iT}^{2+it} \frac{\zeta'(s)}{\zeta(s)} ds\right) \tag{10}$$

now

$$-\int_{\frac{1}{2}+iT}^{2+it} \frac{\zeta'(s)}{\zeta(s)} ds = \log\left(\zeta\left(\frac{1}{2} + iT\right)\right) - \log(\zeta(2+iT))$$

so, remembering that $Im(\log(s)) = arg(s)$ we have:

$$-Im\left(\int_{\frac{1}{2}+iT}^{2+it} \frac{\zeta'(s)}{\zeta(s)} ds\right) = \arg\left(\zeta\left(\frac{1}{2} + iT\right)\right) - \arg(\zeta(2+iT)).$$

The function $\frac{1}{\pi}\arg\left(\zeta\left(\frac{1}{2}+iT\right)\right)$ is sometimes referred to as $S(t)$ and is $\mathcal{O}(\log(T))$, that is also true for $\arg(\zeta(2+iT))$, proof of both can be found in [2] (both are consequences of Lemma 9.4 but only the first is made explicit it Theorem 9.4).

Hence our final result for the value $N(T)$ is:

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \mathcal{O}(\log(T)). \tag{11}$$

□

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Thank you!

We hope this lesson has been beneficial in studying this interesting topic.
For more lessons or demonstrations, visit our website.

References

[1] Walter Dittrich. “On Riemann’s Paper," On the Number of Primes Less Than a Given Magnitude"”. In: *arXiv preprint arXiv:1609.02301* (2016).

[2] Edward Charles Titchmarsh and David Rodney Heath-Brown. *The theory of the Riemann zeta-function*. Oxford university press, 1986.