POSITIVE INCREMENT

Building on the foundations laid by Bernhard Riemann in the 19th century, the **Generalized Riemann Hypothesis** (GRH) extends the famous conjecture about the distribution of prime numbers and their connection to the zeros of the Riemann Zeta Function.

The Generalizations of the Riemann Hypothesis regard a class of functions known as Global L-functions rather than a single one. One can apply similar conditions to these functions and ask if the nontrivial zeros again occur only when $\mathrm{Re}(s) = \frac{1}{2}$.

In particular, the Generalized Riemann Hypothesis focuses on Dirichlet L-functions.

Let's start by defining a Dirichlet character χ:

Definition 1. *A* ***Dirichlet Character modulo*** k*, is an arithmetical function* $\chi(n)$ *(i.e. a real- or complex-valued function defined on the positive integers), such that:*

1. χ *is completely multiplicative, that is to say:*

$$\chi(mn) = \chi(m)\chi(n) \qquad \forall m, n \in \mathbb{N}.$$

2. χ *is periodic of period* k*, that is to say:*

$$\chi(n + k) = \chi(n) \qquad \forall n \in \mathbb{N}.$$

3. $\chi(n) = 0$ *whenever* $(n, k) > 1$*.*

The simplest example of a Dirichlet character is $\chi_1(n)$, the **Principal character modulo k**, defined as:

$$\chi_1(n) = \begin{cases} 1 & \text{if } (n, k) = 1, \\ 0 & \text{if } (n, k) > 1. \end{cases}$$

This obviously satisfies the properties required in definition 1.

It can be proven (see [1], chapter 6) that there are exactly $\varphi(k)$ distinct Dirichlet characters modulo k, where φ is Euler's Totient Function, and that $\chi(n)$ is always a $\varphi(k)$-th root of unity.

In the cited book it's explained how these properties allow us to compute several complete tables of Dirichlet characters modulo k. We propose two examples:

n	1	2	3	4	5
$\chi_1(n)$	1	1	1	1	0
$\chi_2(n)$	1	-1	-1	1	0
$\chi_3(n)$	1	i	$-i$	-1	0
$\chi_4(n)$	1	$-i$	i	-1	0

Table 1: Dirichlet Characters modulo 5.

n	1	2	3	4	5	6	7
$\chi_1(n)$	1	1	1	1	1	1	0
$\chi_2(n)$	1	1	-1	1	-1	-1	0
$\chi_3(n)$	1	ω^2	ω	$-\omega$	$-\omega^2$	-1	0
$\chi_4(n)$	1	ω^2	$-\omega$	$-\omega$	ω^2	1	0
$\chi_5(n)$	1	$-\omega$	ω^2	ω^2	$-\omega$	1	0
$\chi_6(n)$	1	$-\omega$	$-\omega^2$	ω^2	ω	-1	0

Table 2: Dirichlet Characters modulo 7, here $\omega = e^{\frac{i\pi}{3}}$.

Definition 2. *A* ***Dirichlet L-function*** $L(s,\chi)$ *is defined for* $\mathrm{Re}(s) > 1$ *by the series:*

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{1}$$

The Generalized Riemann Hypothesis regards this category of functions, but before stating it, we need to extend these L-functions to the whole complex plane, to do this we define the Hurwitz Zeta Function.

Definition 3. *The* ***Hurwitz Zeta function*** $\zeta(s,a)$*, where* a *is a fixed real number* $0 < a \le 1$ *is defined for* $\mathrm{Re}(s) > 1$ *by the series:*

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}. \tag{2}$$

Remark 1. *It's easy to see that* $\zeta(s,1) = \zeta(s)$.

We can express any Dirichlet L-function in terms of Hurwitz Zeta functions:

Theorem 1. *For any Dirichlet L-function, we have:*

$$L(s,\chi) = k^{-s} \sum_{r=1}^{k} \chi(r)\zeta\left(s, \frac{r}{k}\right) \tag{3}$$

where χ is a Dirichlet character modulo k.

Proof. Write each n from 1 to infinity appearing in the series definition of $L(s,\chi)$ as $n = qk + r$ where $1 \le r \le k$ and $q = 0, 1, 2, \cdots$. Then:

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{r=1}^{k}\sum_{q=0}^{\infty} \frac{\chi(qk+r)}{(qk+r)^s}.$$

Now using the periodicity of Dirichlet characters modulo k:

$$L(s,\chi) = \sum_{r=1}^{k}\sum_{q=0}^{\infty} \frac{\chi(qk+r)}{(qk+r)^s} = \frac{1}{k^s}\sum_{r=1}^{k}\chi(r)\sum_{q=0}^{\infty}\frac{1}{\left(q+\frac{r}{k}\right)^s} = k^{-s}\sum_{r=1}^{k}\chi(r)\zeta\left(s,\frac{r}{k}\right).$$

□

Using theorem 1, we will find an analytic continuation for the L-functions, using an analytic continuation of the Hurwitz Zeta function.

Theorem 2. *The Hurwitz Zeta function satisfies the equation:*

$$\zeta(s,a) = \frac{\Gamma(1-s)}{2\pi i}\int_{-\infty}^{(0,+)} \frac{t^{s-1}e^{at}}{1-e^t}\mathrm{d}t \tag{4}$$

Here the integration contour is C, a loop around the negative real axis; it starts at $-\infty$, encircles the origin once in the positive direction without enclosing any of the points $t = \pm 2\pi i, \pm 4\pi i, \cdots$, and returns to $-\infty$.
t^{-s} has its principal value where t crosses the positive real axis and is continuous.

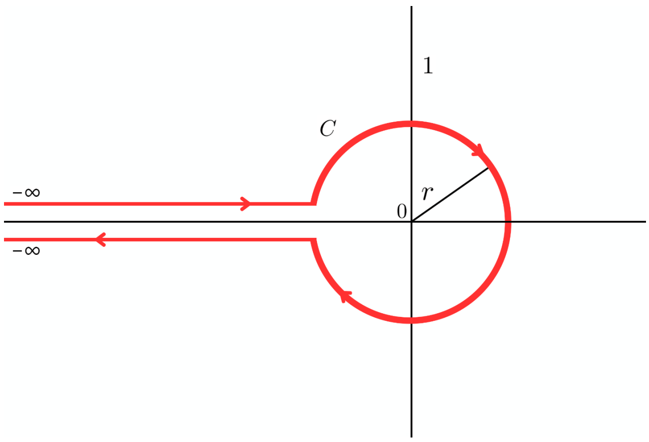

Figure 1: The Contour C

This is a result that is hard to find proven in detail, this is a more complete version of the demonstration from [2].

Proof. Start by considering the formula for the Gamma function known as the Hankel's Loop Integral representation:

$$\frac{1}{\Gamma(s)} = \frac{1}{2\pi i}\int_C e^z z^{-s}\mathrm{d}z$$

which implies:

$$\frac{2\pi i}{\Gamma(s)} = \int_C e^z z^{-s} \mathrm{d}z \tag{5}$$

the contour C is still the one in the picture above.

Fix $v \in \mathbb{C}$ with $\mathrm{Re}(v) > 0$ and substitute $z = vt$ in the integral to obtain:

$$\frac{2\pi i}{\Gamma(s)} = \int_C e^z z^{-s} \mathrm{d}z = \int_C e^{vt} (vt)^{-s} v \mathrm{d}t = v^{1-s} \int_C e^{vt} t^{-s} \mathrm{d}t$$

$$\Downarrow$$

$$\frac{2\pi i}{\Gamma(s)} v^{s-1} = \int_C e^{vt} t^{-s} \mathrm{d}t$$

note that, using Cauchy's Theorem, having chosen $v \in \mathbb{C}$ with $\mathrm{Re}(v) > 0$, the contour can be left unchanged.

Replacing s with $1 - s$ we have:

$$\frac{2\pi i}{\Gamma(1-s)} v^{1-s-1} = \int_C e^{vt} t^{-(1-s)} \mathrm{d}t$$

$$\Downarrow$$

$$v^{-s} = \frac{\Gamma(1-s)}{2\pi i} \int_C e^{vt} t^{s-1} \mathrm{d}t$$

substitute v with $v + n$, where $n \in \mathbb{N}_{\geq 0}$, so that we still have $\mathrm{Re}(v + n) > 0$:

$$(v+n)^{-s} = \frac{\Gamma(1-s)}{2\pi i} \int_C e^{(v+n)t} t^{s-1} \mathrm{d}t.$$

Fix now $z \in \mathbb{C}$ with $|z| \leq 1$.
Restrict the contour C so that any $t \in C$ satisfies $|ze^t| < r < 1$, where r is the radius of the loop around the origin (see Figure 1), once again Cauchy's Theorem ensures that the result does not change. Multiply both sides by z^n:

$$z^n (v+n)^{-s} = z^n \frac{\Gamma(1-s)}{2\pi i} \int_C e^{(v+n)t} t^{s-1} \mathrm{d}t$$

taking the sum from zero to infinity we have:

$$\sum_{n=0}^{\infty} (v+n)^{-s} z^n = \frac{\Gamma(1-s)}{2\pi i} \sum_{n=0}^{\infty} z^n \int_C e^{(v+n)t} t^{s-1} \mathrm{d}t$$

$$\Downarrow$$

$$\sum_{n=0}^{\infty} (v+n)^{-s} z^n = \frac{\Gamma(1-s)}{2\pi i} \int_C t^{s-1} e^{vt} \sum_{n=0}^{\infty} (ze^t)^n \mathrm{d}t$$

we can move the summation under the integral sign because $|ze^t| < r < 1$.
The series on the right hand side is of the geometric kind and $|ze^t| < 1$, therefore we can use the formula: $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$.

We find:

$$\sum_{n=0}^{\infty} (v+n)^{-s} z^n = \frac{\Gamma(1-s)}{2\pi i} \int_C t^{s-1} e^{vt} (1 - ze^t)^{-1} \mathrm{d}t. \tag{6}$$

Remark 2. *The series* $\sum_{n=0}^{\infty}(v+n)^{-s}z^n$ *is sometimes referred to as the function* $\Phi(z,s,v)$ *(see for example [2]).*

Choosing now $z = 1$ and $v = a$ we have:

$$\sum_{n=0}^{\infty}(a+n)^{-s}(1)^n = \frac{\Gamma(1-s)}{2\pi i}\int_C t^{s-1}e^{at}(1-e^t)^{-1}\mathrm{d}t \tag{7}$$

$$\Downarrow$$

$$\sum_{n=0}^{\infty}\frac{1}{(n+a)^s} = \frac{\Gamma(1-s)}{2\pi i}\int_C \frac{t^{s-1}e^{at}}{1-e^t}\mathrm{d}t \tag{8}$$

$$\Downarrow$$

$$\zeta(s,a) = \frac{\Gamma(1-s)}{2\pi i}\int_C \frac{t^{s-1}e^{at}}{1-e^t}\mathrm{d}t.$$

□

Due to the fact that $\Gamma(1-s)$ and the integral are well defined for $\mathrm{Re}(s) \leq 1$, we can use equation 4 to define the Hurwitz function on the whole complex plane.

Definition 4. *If* $\mathrm{Re}(s) \leq 1$*, we define* $\zeta(s,a)$ *by the equation:*

$$\zeta(s,a) = \frac{\Gamma(1-s)}{2\pi i}\int_C \frac{t^{s-1}e^{at}}{1-e^t}\mathrm{d}t.$$

With this definition, the function is analytic for all s *except for a simple pole at* $s = 1$ *with residue* 1.

Remark 3. *Proving that the residue at* $s = 1$ *is* 1 *is fairly simple, it's explained in detail on [1], page 255.*

Definition 5. *If* $\mathrm{Re}(s) \leq 1$*, we define* $L(s,\chi)$ *by the equation:*

$$L(s,\chi) = k^{-s}\sum_{r=1}^{k}\chi(r)\zeta\left(s,\frac{r}{k}\right).$$

Theorem 3.
For the principal character χ_1 *modulo* k*, the L-function* $L(s,\chi_1)$ *is analytic everywhere except for a simple pole at* $s = 1$ *with residue* $\frac{\varphi(k)}{k}$.

If $\chi \neq \chi_1$*,* $L(s,\chi)$ *is an entire function of* s.

Proof. We will use the fact that:

$$\sum_{r \mod k}\chi(r) = \begin{cases} 0 & \text{if } \chi \neq \chi_1, \\ \varphi(k) & \text{if } \chi = \chi_1. \end{cases}$$

Remark 4. *This is also proved in [1].*

We know that $\zeta\left(s, \frac{r}{k}\right)$ has a simple pole at $s = 1$ with residue 1, therefore the function $\chi(r)\zeta\left(s, \frac{r}{k}\right)$ has a simple pole at $s = 1$ with residue $\chi(r)$.

Hence:

$$\begin{aligned} \operatorname{Res}_{s=1} L(s,\chi) &= \lim_{s\to 1}(s-1)L(s,\chi) = \lim_{s\to 1}(s-1)k^{-s}\sum_{r=1}^{k}\chi(r)\zeta\left(s,\frac{r}{k}\right) \\ &= \frac{1}{k}\sum_{r=1}^{k}\chi(r) = \begin{cases} 0 & \text{if } \chi \neq \chi_1, \\ \frac{\varphi(k)}{k} & \text{if } \chi = \chi_1. \end{cases} \end{aligned} \tag{9}$$

□

Now that we have a satisfying definition of these series on the whole complex plane, we can state the Generalized Riemann Hypothesis.

Definition 6. *The **Generalized Riemann Hypothesis** asserts that, for every Dirichlet Character* χ*, if* $L(\chi, s) = 0$ *and* s *is not a negative real number, then* $\operatorname{Re}(s) = \frac{1}{2}$.

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Thank you!

We hope this lesson has been beneficial in studying this interesting topic.
For more lessons or demonstrations, visit our website.

References

[1] Tom M Apostol. *Introduction to analytic number theory*. Springer Science & Business Media, 2013.

[2] Arthur Erdélyi. “Higher transcendental functions”. In: *Higher transcendental functions* (1953), p. 59.