In his now legendary paper "On the Number of Primes Less Than a Given Quantity", Riemann stated his famous hypothesis.

While today we know that the Zeta Function is valuable in a wide range of mathematical topics, its first definition was strictly related to the behavior of prime numbers.

In particular, Riemann defined an unusual Prime Counting Function and proved an integral equation that contains his Zeta Function.

Remark 1. *This lesson explains in detail part of Riemann's Original Paper [2].*

He started by remembering Euler's Product Formula for the Zeta Function:

$$\zeta(s) = \prod_{p} \frac{1}{1 - p^{-s}}. \tag{1}$$

He then defined his Prime Counting Function $F(x)$ as:

$$F(x) = \begin{cases} \text{the number of primes less than } x & \text{when } x \text{ is not a prime} \\ (\text{the number of primes less than } x) + \frac{1}{2} & \text{when } x \text{ is a prime} \end{cases} \tag{2}$$

Remark 2. *Over the years, the Function $F(x)$ originally defined by Riemann has undergone changes in notation and definition; many textbooks prefer not to adhere to the original. Instead, we aim to be as faithful as possible to Riemann's paper.*

This definition implies that, for any k at which there is a jump in the value of $F(x)$ we have:

$$F(k) = \frac{\lim_{x \to 0^+} F(k + x) + \lim_{x \to 0^-} F(k + x)}{2}.$$

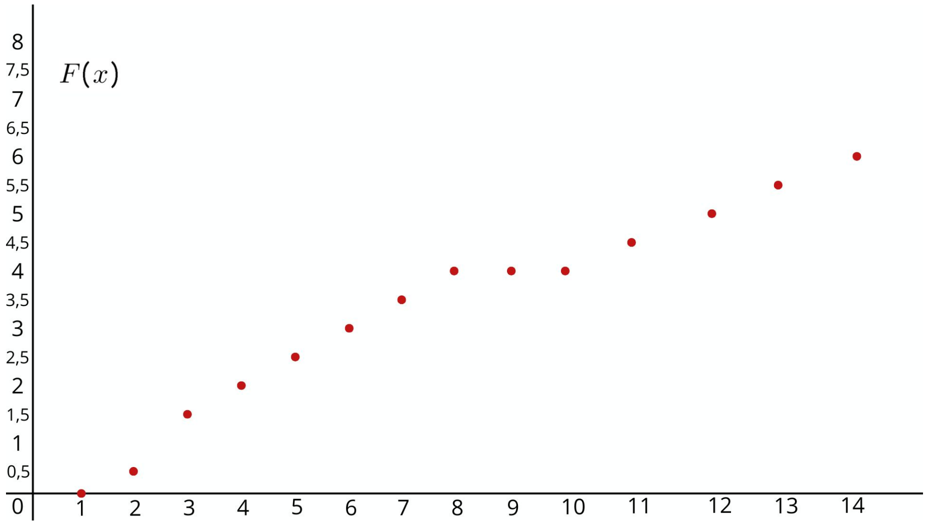

Figure 1: The Riemann Prime Counting Function

By using the Euler Product 1 and and the Taylor expansion for $\log(1+x)$ we obtain the identity:

$$
\begin{aligned}
\log\zeta(s) &= -\sum_p \log\left(1-\frac{1}{p^s}\right) = -\sum_p \log\left(1+\left(-\frac{1}{p^s}\right)\right) \\
&= -\sum_p \sum_{m=1}^{\infty} (-1)^{m+1} \frac{\left(-\frac{1}{p^s}\right)^m}{m} = \sum_p \sum_{m=1}^{\infty} (-1)^{m+2}(-1)^m \frac{1}{mp^{ms}} \\
&= \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}} = \sum_p p^{-s} + \frac{1}{2}\sum_p p^{-2s} + \frac{1}{3}\sum_p p^{-3s} + \cdots
\end{aligned}
\tag{3}
$$

while, using the fact that $\mathrm{Re}(s) > 1$, we have:

$$
s\int_{p^k}^{\infty} x^{-s-1}\mathrm{d}x = \left| s\frac{1}{sx^s} \right|_{p^k}^{\infty} = \frac{1}{p^{ks}} = p^{-ks}.
$$

Hence, equation 3 can be written as:

$$
\begin{aligned}
\log\zeta(s) &= \sum_p p^{-s} + \frac{1}{2}\sum_p p^{-2s} + \cdots \\
&= s\left(\sum_p \int_p^{\infty} x^{-s-1}\mathrm{d}x + \frac{1}{2}\sum_p \int_{p^2}^{\infty} x^{-s-1}\mathrm{d}x + \cdots\right).
\end{aligned}
\tag{4}
$$

Looking at the graph of $F(x)$ (see image 1), one sees that:

$$
\begin{aligned}
&\int_1^{\infty} F(x)x^{-s-1}\mathrm{d}x = \int_2^3 x^{-s-1}\mathrm{d}x + 2\int_3^5 x^{-s-1}\mathrm{d}x + 3\int_5^7 x^{-s-1}\mathrm{d}x + \cdots \\
&= \sum_{k=1}^{\infty} k\int_{p_k}^{p_{k+1}} x^{-s-1}\mathrm{d}x
\end{aligned}
\tag{5}
$$

where p_k is the $k-$th prime number. Notice that:

$$\int_{p_k}^{p_{k+1}} x^{-s-1}\mathrm{d}x = \int_{p_k}^{\infty} x^{-s-1}\mathrm{d}x - \int_{p_{k+1}}^{\infty} x^{-s-1}\mathrm{d}x.$$

Therefore:

$$\begin{aligned}\sum_{k=1}^{\infty} k \int_{p_k}^{p_{k+1}} x^{-s-1}\mathrm{d}x &= \int_2^{\infty} x^{-s-1}\mathrm{d}x - \int_3^{\infty} x^{-s-1}\mathrm{d}x + 2\int_3^{\infty} x^{-s-1} - 2\int_5^{\infty} x^{-s-1}\mathrm{d}x + \cdots \\ &= \int_2^{\infty} x^{-s-1}\mathrm{d}x + \int_3^{\infty} x^{-s-1}\mathrm{d}x + \int_5^{\infty} x^{-s-1}\mathrm{d}x + \cdots = \sum_p \int_p^{\infty} x^{-s-1}\mathrm{d}x\end{aligned} \tag{6}$$

which is exactly the first term of equation 4.

One can see with the same computations that:

$$\begin{aligned}\frac{1}{2}\int_1^{\infty} F(x^{\frac{1}{2}})x^{-s-1}\mathrm{d}x &= \frac{1}{2}\left[\int_4^{9} x^{-s-1}\mathrm{d}x + 2\int_9^{25} x^{-s-1}\mathrm{d}x + 3\int_{25}^{49} x^{-s-1}\mathrm{d}x + \cdots\right] \\ &= \frac{1}{2}\sum_{k=1}^{\infty} k\int_{p_k^2}^{p_{k+1}^2} x^{-s-1}\mathrm{d}x = \frac{1}{2}\sum_p \int_{p^2}^{\infty} x^{-s-1}\mathrm{d}x.\end{aligned} \tag{7}$$

which is the second term of equation 4.

Similar equations can be found for $\frac{1}{3}F(x^{\frac{1}{3}})$, $\frac{1}{4}F(x^{\frac{1}{4}})$,$\frac{1}{5}F(x^{\frac{1}{5}})$,$\cdots$.

Therefore calling

$$f(x) := F(x) + \frac{1}{2}F(x^{\frac{1}{2}}) + \frac{1}{3}F(x^{\frac{1}{3}}) + \cdots$$

Riemann obtains the equation:

$$\begin{gathered}\log\zeta(s) = s\int_1^{\infty} f(x)x^{-s-1}\mathrm{d}x \\ \Downarrow \\ \frac{\log\zeta(s)}{s} = \int_1^{\infty} f(x)x^{-s-1}\mathrm{d}x \\ \Downarrow \\ \frac{\log\zeta(s)}{s} = \int_0^{\infty} f(x)x^{-s-1}\mathrm{d}x.\end{gathered} \tag{8}$$

This equation is valid for each complex value $s = a + ib$ for which $a > 1$.

Remark 3. *In his original article, Riemann never writes this integral from* 0 *to* ∞*, but this last equivalence is obvious due to the definition of* f *and will be useful for the next computation.*

After obtaining equation 8 Riemann notices that, if, for $s = a + ib$, $a > 1$ and $h(x)$ a real function, a function $g(s)$ satisfies

$$g(s) = \int_0^\infty h(x)x^{-s} \mathrm{d}\log x$$

Remark 4. *This is an example of a Riemann-Stieltjes integral, for details about this theory we recommend the book "The Stieltjes integral" by G.Convertito and D. Cruz-Uribe [1].*
To those only familiar with Measure theory, we can say roughly speaking that this is essentially using $\log x$ *as a measure.*

Then the function $g(s)$ can be decomposed as:

$$\begin{aligned}
&g(s) = \int_0^\infty h(x)x^{-s} \mathrm{d}\log x = \int_0^\infty h(x)e^{(-a-ib)\log x} \mathrm{d}\log x \\
&= \int_0^\infty h(x)x^{-a}e^{-ib\log x} \mathrm{d}\log x = \int_0^\infty h(x)x^{-a}(\cos(-b\log x) + i\sin(-b\log x))\mathrm{d}\log x \\
&= \int_0^\infty h(x)x^{-a}\cos(b\log x)\mathrm{d}\log x - i\int_0^\infty h(x)x^{-a}\sin(b\log x)\mathrm{d}\log x.
\end{aligned} \tag{9}$$

That is to say:

$$g(a + ib) = g_1(b) + ig_2(b)$$

where

$$g_1(b) = \int_0^\infty h(x)x^{-a}\cos(b\log x)\mathrm{d}\log x$$

$$ig_2(b) = -i\int_0^\infty h(x)x^{-a}\sin(b\log x)\mathrm{d}\log x.$$

Proceed by multiplying both equations by:

$$(\cos(b\log y) + i\sin(b\log y))\mathrm{d}b = e^{ib\log y}\mathrm{d}b$$

and integrating from $-\infty$ to $+\infty$ to obtain, on the right hand side:

$$\int_{-\infty}^{+\infty}\int_0^\infty h(x)x^{-a}e^{-ib\log x}e^{ib\log y}\mathrm{d}\log x\mathrm{d}b.$$

A known property of the Riemann-Stieltjes integral is that, for $f(x)$ bounded on the integration interval, $g(x)$ monotonically increasing and $g'(x)$ Riemann integrable, we have:

$$\int_a^b f(x)dg(x) = \int_a^b f(x)g'(x)\mathrm{d}x$$

so in our case, $f(x) = h(x)x^{-a}e^{-ib\log x}e^{ib\log y}$, $g(x) = \log x$ and:

$$\int_{-\infty}^{+\infty}\int_0^\infty h(x)x^{-a}e^{-ib\log x}e^{ib\log y}\mathrm{d}\log x\mathrm{d}b = \int_{-\infty}^{+\infty}\int_0^\infty h(x)x^{-a-1}e^{-ib\log x}e^{ib\log y}\mathrm{d}x\mathrm{d}b$$

now changing variable to $t = \log x \Rightarrow x = e^t$:

$$\int_{-\infty}^{+\infty}\int_{0}^{\infty} h(x)x^{-a-1}e^{-ib\log x}e^{ib\log y}\mathrm{d}x\mathrm{d}b = \int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} h\left(e^t\right)e^{(-a-1)t}e^{-ibt}e^{ib\log y}e^t\mathrm{d}t\mathrm{d}b$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} h\left(e^t\right)e^{-at}e^{i(\log y-t)\cdot b}\mathrm{d}t\mathrm{d}b = 2\pi\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} h\left(e^t\right)e^{-at}e^{i(\log y-t)\cdot b}\mathrm{d}t\mathrm{d}b\right]$$
$$= 2\pi h\left(e^{\log y}\right)e^{(-a)\log y} = 2\pi h(y)y^{-a} \tag{10}$$

where we used Fourier's inversion theorem in the last passage.

We have therefore proven that:

$$\int_{-\infty}^{+\infty} g(s)e^{ib\log y}\mathrm{d}b = 2\pi h(y)y^{-a}$$

If we multiply both sides by iy^a we obtain:

$$\int_{-\infty}^{+\infty} g(s)e^{ib\log y}y^a i\mathrm{d}b = 2\pi ih(y)$$
$$\Downarrow$$
$$\int_{-\infty}^{+\infty} g(s)y^{a+ib}i\mathrm{d}b = 2\pi ih(y)$$
$$\Downarrow$$
$$\int_{a-i\infty}^{a+i\infty} g(s)y^s\mathrm{d}s = 2\pi ih(y).$$

The functions $f(x)$ and $\log(s)$ satisfy the same properties required for $h(x)$ and $g(s)$ respectively. Therefore, applying this result we conclude:

$$f(y) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\log\zeta(s)}{s}y^s\mathrm{d}s. \tag{11}$$

POSITIVE INCREMENT

Thank you!

We hope this lesson has been beneficial in studying this interesting topic.
For more lessons or demonstrations, visit our website.

References

[1] Gregory Convertito and David Cruz-Uribe. *The Stieltjes Integral*. Chapman and Hall/CRC, 2023.

[2] Bernhard Riemann. "On the number of prime numbers less than a given quantity (ueber die anzahl der primzahlen unter einer gegebenen grösse)". In: *Monatsberichte der Berliner Akademie* (1859).