Various approaches have tried to tackle the Riemann Hypothesis, many of which come from fields outside of analytic number theory.
A notable example is the de Bruijn-Newman constant.

This topic was first introduced by de Bruijn in his paper on "The Roots of Trigonometric Integrals", where comments related to the Riemann Hypothesis appeared only in the final section. Newman later expanded on this theory in his article about "Fourier Transforms with Only Real Zeros".
While this paper focused more on the Riemann Hypothesis, it is also noteworthy to highlight the connection between the zeros of Fourier transforms and areas such as statistical mechanics and quantum field theory.

Newman proposed a conjecture that would remain unresolved for nearly 50 years.

Remark 1. *The inspiration for this lesson comes from the articles written by de Bruijn, Newman, and Rodgers-Tao ([1],[2],[3]). Additionally, you can find a brief introduction to the topic on Terence Tao's blog. The post includes a rough outline of the proof of the conjecture.*

To understand the de Bruijn-Newman constant, we first derive the function H_0 from the Riemann ξ function.

Remember the definition of the Riemann ξ function:

$$\xi(s) := \frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{1}$$

where $\Gamma(s)$ is the Gamma Function.

Remark 2. *With this expression, the function is only defined for $Re(s) > 1$, however, it was already known to Riemann that it is possible to extend it to the entire complex plane. Details of this process can be found on our lesson explaining Riemann's original manuscript.*

We proceed by manipulating this expression of the ξ function:

First, remember the fact that $s\Gamma(s) = \Gamma(s+1)$ and write:

$$\frac{s(s-1)}{2}\Gamma\left(\frac{s}{2}\right) = (s-1)\Gamma\left(\frac{s}{2}+1\right) = (s+2-3)\Gamma\left(\frac{s}{2}+1\right) = 2\left(\frac{s}{2}+1\right)\Gamma\left(\frac{s}{2}+1\right) - 3\Gamma\left(\frac{s}{2}+1\right)$$

$$\Downarrow$$

$$\frac{s(s-1)}{2}\Gamma\left(\frac{s}{2}\right) = 2\Gamma\left(\frac{s+4}{2}\right) - 3\Gamma\left(\frac{s+2}{2}\right).$$

Hence, using the integral expression of the Gamma Function $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$ and the definition of the Riemann Zeta, $\zeta(s) = \sum_{n=1}^\infty n^{-s}$, we have:

$$\begin{aligned} \xi(s) &= \zeta(s)\pi^{-\frac{s}{2}}\left(2\Gamma\left(\frac{s+4}{2}\right) - 3\Gamma\left(\frac{s+2}{2}\right)\right) \\ &= 2\zeta(s)\pi^{-\frac{s}{2}}\int_0^\infty e^{-t}t^{\frac{s+4}{2}-1}dt - 3\zeta(s)\pi^{-\frac{s}{2}}\int_0^\infty e^{-t}t^{\frac{s+2}{2}-1}dt \\ &= \sum_{n=1}^\infty 2\pi^{-\frac{s}{2}}n^{-s}\int_0^\infty e^{-t}t^{\frac{s+4}{2}-1}dt - 3\pi^{-\frac{s}{2}}n^{-s}\int_0^\infty e^{-t}t^{\frac{s+2}{2}-1}dt. \end{aligned} \tag{2}$$

Scale the variable $t \to \pi n^2 t$ to write:

$$\int_0^\infty e^{-t}t^{\frac{s+4}{2}-1}dt = \int_0^\infty e^{-\pi n^2 t}(\pi n^2 t)^{\frac{s+4}{2}-1}\pi n^2 dt = (\pi n^2)^{\frac{s+4}{2}}\int_0^\infty e^{-\pi n^2 t}t^{\frac{s+4}{2}-1}dt$$

and

$$\int_0^\infty e^{-t}t^{\frac{s+2}{2}-1}dt = \int_0^\infty e^{-\pi n^2 t}(\pi n^2 t)^{\frac{s+2}{2}-1}\pi n^2 dt = (\pi n^2)^{\frac{s+2}{2}}\int_0^\infty e^{-\pi n^2 t}t^{\frac{s+2}{2}-1}dt.$$

Equation 2 can therefore be written as:

$$\xi(s) = \sum_{n=1}^\infty 2\pi^{-\frac{s}{2}}n^{-s}(\pi n^2)^{\frac{s+4}{2}}\int_0^\infty e^{-\pi n^2 t}t^{\frac{s+4}{2}-1}dt - 3\pi^{-\frac{s}{2}}n^{-s}(\pi n^2)^{\frac{s+2}{2}}\int_0^\infty e^{-\pi n^2 t}t^{\frac{s+2}{2}-1}dt$$

$$\Downarrow$$

$$\xi(s) = \sum_{n=1}^\infty 2\pi^2 n^4\int_0^\infty e^{-\pi n^2 t}t^{\frac{s+4}{2}-1} - 3\pi n^2 e^{-\pi n^2 t}t^{\frac{s+2}{2}-1}dt.$$

We can apply Fubini's Theorem to switch the integral and the series, so:

$$\xi(s) = \int_0^\infty \sum_{n=1}^\infty 2\pi^2 n^4 e^{-\pi n^2 t}t^{\frac{s+4}{2}-1} - 3\pi n^2 e^{-\pi n^2 t}t^{\frac{s+2}{2}-1}dt.$$

Change variable to $t = e^{4u}$ to obtain:

$$\begin{aligned} \xi(s) &= \int_{\mathbb{R}}\left(\sum_{n=1}^\infty 2\pi^2 n^4 e^{-\pi n^2 e^{4u}}e^{4u\left(\frac{s+4}{2}-1\right)} - 3\pi n^2 e^{-\pi n^2 e^{4u}}e^{4u\left(\frac{s+2}{2}-1\right)}\right)4e^{4u}du \\ &= 4\int_{\mathbb{R}}\sum_{n=1}^\infty 2\pi^2 n^4 e^{8u}e^{2su-\pi n^2 e^{4u}} - 3\pi n^2 e^{4u}e^{2su-\pi n^2 e^{4u}}du \end{aligned} \tag{3}$$

$$\Downarrow$$

$$\xi(s) = 4\int_{\mathbb{R}} \sum_{n=1}^{\infty} \left(2\pi^2 n^4 e^{8u} - 3\pi n^2 e^{4u}\right) e^{2su - \pi n^2 e^{4u}} du. \tag{4}$$

Define $H_0(z)$ the function:

$$H_0(z) := \frac{1}{8}\xi\left(\frac{1}{2} + \frac{iz}{2}\right) \tag{5}$$

equation 4 implies that, at least for $Im(z) > 1$:

$$H_0(z) = \frac{1}{2}\int_{\mathbb{R}} \Phi(u) e^{izu} du \tag{6}$$

where we denoted $\Phi(u)$ the function:

$$\Phi(u) := \sum_{n=1}^{\infty} \left(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}\right) e^{-\pi n^2 e^{4u}}.$$

The function Φ can be verified to be rapidly decreasing both as $u \to \infty$ and $u \to -\infty$ with the former faster than any exponential. Therefore, H_0 extends holomorphically to the upper half plane.

We need now to recall two basic properties of Fourier Transformation:

First, it is easily verified that

$$\mathcal{F}[e^{-\pi x^2}](y) := \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i y x} dx = e^{-\pi y^2}$$

that is to say, the function $e^{-\pi x^2}$ is its own Fourier Transform.

Second, the operator $2\pi x - \frac{d}{dx}$ interacts with the Fourier Transform by the identity:

$$\mathcal{F}\left[\left(2\pi x - \frac{d}{dx}\right) f\right](y) = -i\left(2\pi y - \frac{d}{dy}\right)\mathcal{F}[f](y). \tag{7}$$

Notice that $(-i)^4 = 1$, therefore applying the operator 4 times eliminates the $-i$ factor.

Remark 3. *The operator $2\pi x - \frac{d}{dx}$ is often referred to as the "Creation operator" due to its use in quantum dynamics. Equation 7 shows that in some sense the Creation operator and the Fourier Transform commute.*

We will combine these properties:

Compute, using 7, the Fourier transform of the function obtained by applying the operator $2\pi x - \frac{d}{dx}$ on $f = e^{-\pi x^2}$ 4 times, explicitly:

$$\mathcal{F}\left[\left(2\pi x - \frac{d}{dx}\right)^4 e^{-\pi x^2}\right](y) = \left(2\pi y - \frac{d}{dy}\right)^4 \mathcal{F}\left[e^{-\pi x^2}\right](y) = \left(2\pi y - \frac{d}{dy}\right)^4 e^{-\pi y^2}.$$

Which proves that the function $\left(2\pi x - \frac{d}{dx}\right)^4 e^{-\pi x^2}$ is also its own Fourier Transform.

One can compute:

$$\left(2\pi x - \frac{d}{dx}\right)^4 e^{-\pi x^2} = 128\pi^2(2\pi^2 x^4 - 3\pi x^2)e^{-\pi x^2} + 48\pi^2 e^{-\pi x^2}.$$

Using the linearity of the Fourier Transform, we can deduce that:

$$128\pi^2(2\pi^2 x^4 - 3\pi x^2)e^{-\pi x^2} + 48\pi^2 e^{-\pi x^2} - 48\pi^2 e^{-\pi x^2} = 128\pi^2(2\pi^2 x^4 - 3\pi x^2)e^{-\pi x^2}$$

is also its own Fourier Transform, and therefore, so is $(2\pi^2 x^4 - 3\pi x^2)e^{-\pi x^2}$.

Remember now the scaling property of the Fourier transform:

$$\mathcal{F}\left[f(ax)\right](y) = \frac{1}{|a|}\mathcal{F}\left[f(x)\right]\left(\frac{y}{a}\right). \tag{8}$$

This implies that scaling $x \to xe^{2u}$ we have:

$$(2\pi^2 x^4 - 3\pi x^2)e^{-\pi x^2} \to (2\pi^2 x^4 e^{8u} - 3\pi x^2 e^{4u})e^{-\pi x^2 e^{4u}}$$

$$\Downarrow$$

$$\mathcal{F}\left[(2\pi^2 x^4 e^{8u} - 3\pi x^2 e^{4u})e^{-\pi x^2 e^{4u}}\right](y) = \frac{1}{e^{2u}}\mathcal{F}\left[(2\pi^2 x^4 - 3\pi x^2)e^{-\pi x^2}\right]\left(\frac{y}{e^{2u}}\right) = \frac{1}{e^{2u}}$$

$$\Downarrow$$

$$\mathcal{F}\left[(2\pi^2 x^4 e^{8u} - 3\pi x^2 e^{4u})e^{-\pi x^2 e^{4u}}\right](y) = \frac{1}{e^{2u}}\left[\left(2\pi^2\left(\frac{y}{e^{2u}}\right)^4 - 3\pi\left(\frac{y}{e^{2u}}\right)^2\right)e^{-\pi\left(\frac{y}{e^{2u}}\right)^2}\right]$$

$$\Downarrow$$

$$\mathcal{F}\left[(2\pi^2 x^4 e^{8u} - 3\pi x^2 e^{4u})e^{-\pi x^2 e^{4u}}\right](y) = \frac{1}{e^{u}}\left[\left(2\pi^2 y^4 e^{-9u} - 3\pi y^2 e^{-5u}\right)e^{-\pi y^2 e^{-4u}}\right]$$

multiplying everything by e^u and using the linearity of the Fourier Transform, we conclude:

$$\mathcal{F}\left[(2\pi^2 x^4 e^{9u} - 3\pi x^2 e^{5u})e^{-\pi x^2 e^{4u}}\right](y) = \left[\left(2\pi^2 y^4 e^{-9u} - 3\pi y^2 e^{-5u}\right)e^{-\pi y^2 e^{-4u}}\right]. \tag{9}$$

Remember Poisson's summation formula, it states that, for $f : \mathbb{R} \to \mathbb{C}$ a Schwartz function, if $\mathcal{F}[f](y)$ is its Fourier Transform, then:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \mathcal{F}[f](m). \tag{10}$$

Equation 9 expresses $\mathcal{F}\left[(2\pi^2 x^4 e^{9u} - 3\pi x^2 e^{5u})e^{-\pi x^2 e^{4u}}\right](y)$.
Therefore, applying 10:

$$\sum_{n\in\mathbb{Z}}(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u})e^{-\pi n^2 e^{4u}} = \sum_{m\in\mathbb{Z}}\left(2\pi^2 m^4 e^{-9u} - 3\pi m^2 e^{-5u}\right)e^{-\pi m^2 e^{-4u}}.$$

Both of these sums are symmetric and both vanish for $n = m = 0$. Therefore:

$$\sum_{n\in\mathbb{Z}}(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u})e^{-\pi n^2 e^{4u}} = 2\sum_{n=1}^{\infty}(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u})e^{-\pi n^2 e^{4u}} = 2\Phi(u)$$

and

$$\sum_{m\in\mathbb{Z}}\left(2\pi^2 m^4 e^{-9u} - 3\pi m^2 e^{-5u}\right)e^{-\pi m^2 e^{-4u}} = 2\sum_{m=1}^{\infty}\left(2\pi^2 m^4 e^{-9u} - 3\pi m^2 e^{-5u}\right)e^{-\pi m^2 e^{-4u}} = 2\Phi(-u)$$

which brings us to the crucial functional equation:

$$\Phi(u) = \Phi(-u). \tag{11}$$

This implies that both Φ and H_0 are even function (and therefore we have extended H_0 to an entire function).

Using this symmetry we can rewrite equation 6:

$$\begin{aligned}
H_0(z) &= \frac{1}{2}\int_{\mathbb{R}}\Phi(u)e^{izu}du = \frac{1}{2}\left[\int_0^{+\infty}\Phi(u)e^{izu}du + \int_{-\infty}^{0}\Phi(u)e^{izu}du\right] \\
&= \frac{1}{2}\left[\int_0^{+\infty}\Phi(u)e^{izu}du + \int_{+\infty}^{0}\Phi(-u)e^{-izu}(-du)\right] \\
&= \frac{1}{2}\left[\int_0^{+\infty}\Phi(u)e^{izu}du + \int_0^{\infty}\Phi(u)e^{-izu}du\right] \\
&= \frac{1}{2}\int_0^{+\infty}\Phi(u)(e^{izu} + e^{-izu})du \\
&= \frac{1}{2}\int_0^{+\infty}\Phi(u)(2\cos(zu))du
\end{aligned} \tag{12}$$

$$\Downarrow$$

$$H_0(z) = \int_0^{+\infty}\Phi(u)\cos(zu)du. \tag{13}$$

This is a convergent expression for the entire function $H_0(z)$ for all complex z.

Given that Φ is even and real-valued on $\mathbb{R}$, $H_0(z)$ is even and satisfies the functional equation $H_0(\overline{z}) = \overline{H_0(z)}$.
Remember that we defined $H_0(z) = \frac{1}{8}\xi\left(\frac{1}{2} + \frac{iz}{2}\right)$ and therefore:

The Riemann Hypothesis is equivalent to the claim that all the zeroes of $H_0(z)$ are real.

Remark 4. *This formulation of the Hypothesis might seem unusual, but it's actually much closer to Riemann's original statement. For details, check our lesson about the article "On the number of primes less than a given quantity".*

In his 1950 article "The roots of trigonometric integrals" [1], N.G. de Bruijn introduced a family of deformations of $H_0(z)$ defined as:

$$H_t : \mathbb{C} \to \mathbb{C} \qquad H_t(z) := \int_0^{\infty} e^{tu^2}\Phi(u)\cos(zu)du.$$

Remark 5. *In his blog post, Terrence Tao remarks the PDE perspective, interpreting H_t as the evolution of H_0 under the backwards heat equation $\partial_t H_t(z) = -\partial_{zz} H_t(z)$. This point of view is crucial in the demonstration of the conjecture.*

Just like H_0, the H_t are all even entire functions that obey the functional equation $H_t(\overline{z}) = \overline{H_t(z)}$, therefore we can ask an analogue of the Riemann Hypothesis for each such function, namely whether all the zeros of H_t are real.

De Bruijn also showed that satisfying the hypothesis is a monotone property in t, that is to say: if H_t has all real zeroes for some t, then for any $t^{'} \geq t$, $H_{t'}$ also has all real zeroes.

C.M. Newman proved something stronger in his 1976 article "Fourier Transforms with only real zeroes" [2]. He demonstrated that there exists a finite constant $\Lambda \leq \frac{1}{2}$, now known as the **de Bruijn-Newman constant**, with the property that H_t **has all zeroes real if and only if** $t \geq \Delta$**.** Thus the Riemann Hypothesis is equivalent to the inequality $\Delta \leq 0$.

Newman also conjectured that $\Delta \geq 0$; in his words, this conjecture would mean that if the Riemann Hypothesis is true, then it is only "barely so", in that it is disrupted by applying heat flow for even an arbitrary small amount of time.

In their 2018 article "The De Bruijn-Newman constant is non-negative" [3] B. Rodgers and T.Tao have proven this decades old conjecture, showing that indeed $\Delta \geq 0$.

In conclusion: **The Riemann Hypothesis is true** $\iff$ $\Delta = 0$**.**

Thank you!

We hope this lesson has been beneficial in studying this interesting topic.
For more lessons or demonstrations, visit our website.

References

[1] Nicolaas G de Bruijn. "The roots of trigonometric integrals". In: (1950).

[2] Charles M Newman. "Fourier transforms with only real zeros". In: *Proceedings of the American Mathematical Society* 61.2 (1976), pp. 245–251.

[3] Brad Rodgers and Terence Tao. "The de Bruijn–Newman constant is non-negative". In: *Forum of Mathematics, Pi*. Vol. 8. Cambridge University Press. 2020, e6.