The primary focus of this lesson is the **Möbius function** $\mu(n)$. Although it first appeared implicitly in some of Euler's works, it was systematically defined and studied by August Ferdinand Möbius in 1832 [6]. For a deeper exploration of the history and significance of this function, we suggest referring to [4].

Rather than reviewing the original article, we will define the function and prove its main properties. This will allow us to explain its connection to the Riemann Zeta function and the Riemann Hypothesis.

1 Basic properties

Definition 1. *The **Möbius function** is defined as:*

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes} \\ 0 & \text{if } n \text{ is divisible by a square} > 1 \end{cases} \tag{1}$$

note that either n contains at least one squared prime in its factorization or it contains all distinct primes, hence the function is defined for all $n \in \mathbb{N}_{\geq 1}$.

Remark 1.1. *The Möbius function belongs to a particular category called **"arithmetical functions"**. These are a central topic in Analytic Number Theory; a complete introduction to the subject can be found in Apostol's book [1].*

Theorem 1. *We have:*

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases} \tag{2}$$

Proof. The formula is obviously true for $n = 1$, assume $n > 1$ and therefore $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$. By definition of $\mu(n)$, in the sum $\sum_{d|n} \mu(d)$ the only nonzero terms arise when $d = 1$ and when d is a product of distinct primes, therefore:

$$\sum_{d|n} \mu(d) = \mu(1) + \mu(p_1) + \cdots + \mu(p_k) + \mu(p_1 p_2) + \cdots + \mu(p_{k-1} p_k) + \cdots + \mu(p_1 p_2 \cdots p_k).$$

The first terms after $\mu(1)$ are all the possible combinations of k factors taken one at the time, the second terms are all the possible combinations of k factors taken two at the time and so on, in other words:

$$\sum_{d|n}\mu(d) = 1 + \binom{k}{1}(-1) + \binom{k}{2}(-1)^2 + \cdots + \binom{k}{k}(-1)^k.$$

Therefore, remembering that

$$\sum_{m=0}^{k}\binom{k}{m}a^{k-m}b^m = (a+b)^k$$

we have:

$$\sum_{d|n}\mu(d) = \sum_{m=0}^{k}\binom{k}{m}(-1)^m = \sum_{m=0}^{k}\binom{k}{m}1^{k-m}(-1)^m = (1-1)^k = 0.$$

□

Theorem 2. *The Möbius function is multiplicative, that is to say:*

$$\mu(ab) = \mu(a)\mu(b)$$

whenever $a, b \in \mathbb{N}$ are coprime.

Proof. Consider two coprime numbers $a, b \in \mathbb{N}$, without loss of generality, suppose $a \geq b$.

We proceed by induction on ab:
If $ab = 1$, then $\mu(ab) = 1 = \mu(a)\mu(b)$ (as $a = b = 1$).

Otherwise, $ab > 1$ and Theorem 1 implies:

$$0 = \sum_{d|ab}\mu(d) = \mu(ab) + \sum_{d|ab; d<ab}\mu(d).$$

When $d < ab$ the inductive hypothesis assures that the Theorem is valid. Hence:

$$0 = \mu(ab) + \sum_{d|ab; d<ab}\mu(d) = \mu(ab) + \sum_{d|a; d'|b}\mu(d)\mu(d') - \mu(a)\mu(b)$$

$$= \mu(ab) - \mu(a)\mu(b) + \sum_{d|a}\mu(d)\sum_{d|b}\mu(d') = \mu(ab) - \mu(a)\mu(b) + 0.$$

In conclusion:

$$\mu(ab) = \mu(a)\mu(b).$$

□

2 Relation to $\zeta(s)$

This section is a simplified version of the ones contained in [3] and [7] regarding this topic, we chose not to prolong the text with unneeded results. For a broader picture, both the cited sources are valid.

Remark 2.1. *We will use the classic notation $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$, therefore $\sigma = Re(s)$, $t = Im(s)$.*

Theorem 3. *If s is a complex number with $\sigma > 1$, we have:*

$$\sum_{n=1}^{\infty} \frac{\mu(s)}{n^s} = \frac{1}{\zeta(s)} \tag{3}$$

Proof. This proof requires the use of the Euler product for the Riemann zeta function, that is:

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} \tag{4}$$

where the product runs over all prime numbers p and $\sigma > 1$. The proof of this classical result can be found on our site.

Equation 4 implies that:

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - p^{-s}\right) = \prod_p \left(1 - \frac{1}{p^s}\right) = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\cdots$$

Let's compute this product step by step:

Firstly, infinite times the product of 1 gives us 1:

$$\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\cdots = 1 + \cdots$$

then we have the product of 1 times a negative fraction of a prime times infinite times 1, which leaves us only with the fraction; this happens for every prime (included 2) and therefore we have:

$$\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\cdots = 1 + \sum_p \frac{-1}{p^s} + \cdots.$$

We then find the same product, except this time with two negative fraction of primes; once again this happens for every prime, hence:

$$\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\cdots = 1 + \sum_p \frac{-1}{p^s} + \sum_{n=p_1p_2} \frac{-1}{p_1^s}\frac{-1}{p_2^s} + \cdots.$$

Iterating this process we have:

$$\prod_p \left(1 - \frac{1}{p^s}\right) = 1 + \sum_p \frac{-1}{p^s} + \sum_{n=p_1p_2} \frac{-1}{p_1^s}\frac{-1}{p_2^s} + \sum_{n=p_1p_2p_3} \frac{-1}{p_1^s}\frac{-1}{p_2^s}\frac{-1}{p_3^s} + \cdots$$

but $\frac{-1}{p_1^s}\frac{-1}{p_2^s} = \frac{1}{n^s}$ and $\frac{-1}{p_1^s}\frac{-1}{p_2^s}\frac{-1}{p_3^s} = \frac{-1}{n^s}$ therefore:

$$\prod_p \left(1 - \frac{1}{p^s}\right) = 1 + \sum_{n=p} \frac{-1}{n^s} + \sum_{n=p_1p_2} \frac{1}{n^s} + \sum_{n=p_1p_2p_3} \frac{-1}{n^s} + \cdots \tag{5}$$

which is exactly

$$\prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(s)}{n^s}. \tag{6}$$

Notice that when n is divisible by a squared prime, it doesn't appear in 5, indeed in the series 6 those numbers have coefficient 0.

□

Definition 2. *If $x \geq 1$ we define the **Mertens' Function** as:*

$$M(x) = \sum_{n \leq x} \mu(n). \tag{7}$$

It's a known fact that:

$$\frac{1}{\zeta(s)} = \int_0^\infty x^{-s} dM(x) \qquad \text{For } \sigma > 1. \tag{8}$$

This is an example of a Riemann-Stieltjes integral, for details about this theory we recommend the book "The Stieltjes integral" by G.Convertito and D. Cruz-Uribe [2].
Knowing the basis of this theory, equation 8 should be clear.
To those only familiar with Measure theory, we can say roughly speaking that this is essentially using $M(x)$ as the measure. Looking at the definition of this function and Theorem 3, this equation should come to no surprise.

The Riemann-Stieltjes integral allows for integration by parts, so we can write equation 8 as:

$$\begin{aligned} \frac{1}{\zeta(s)} &= \int_0^\infty x^{-s} dM(x) = x^{-s}M(x)|_0^\infty - \int_0^\infty M(x)(-sx^{-s-1})dx \\ &= \lim_{x\to\infty}\left[x^{-s}M(x)\right] + s\int_0^\infty M(x)x^{-s-1}dx = s\int_0^\infty M(x)x^{-s-1}dx \end{aligned} \tag{9}$$

We have therefore proven a crucial formula to understand the connection between the Mertens function and the Riemann Hypothesis:

$$\frac{1}{\zeta(s)} = s\int_0^\infty M(x)x^{-s-1}dx. \tag{10}$$

Convergence here is assured by the obvious equality $|M(x)| \leq x$, it implies that $\left|x^{-s}M(x)\right| \leq \left|x^{-s+1}\right| \to 0$ as $x \to \infty$ and $\int_0^\infty M(x)x^{-s-1}dx$ converges, both provided that $\sigma > 1$.

Notice that if $M(x)$ grows less rapidly than x^a for some $a > 0$, then this integral converges for all s in the half-plane $Re(a - s) < 0 \iff Re(s) > a$, therefore by analytic continuation the function $\frac{1}{\zeta(s)}$ is analytic in this half plane.

This argument brings us to two important conclusion:

Firstly, we definitely know that $\frac{1}{\zeta(s)}$ has poles on the line $\sigma = \frac{1}{2}$ and therefore this formula cannot give us an analytic continuation there, i.e.

$M(x)$ cannot grow less rapidly than x^a for any $a < \frac{1}{2}$.

On the other hand if we were to prove that $M(x)$ grows less rapidly then $x^{\frac{1}{2}+\epsilon}$ for all $\epsilon > 0$, then the function $\frac{1}{\zeta(s)}$ would be confirmed analytic for $\sigma > \frac{1}{2}$, that is to say :

$M(x) = \mathcal{O}\left(x^{\frac{1}{2}+\epsilon}\right)$ for all $\epsilon > 0$ implies the Riemann Hypothesis.

It was actually proven by J.E. Littlewood in [5] that this condition is also necessary to the Riemann Hypothesis, in other words:

Theorem 4. *The Riemann Hypothesis is equivalent to the statement that for every $\epsilon > 0$ the function $M(x)x^{-\frac{1}{2}-\epsilon}$ approaches zero as $x \to 0$.*

This result is very important, we shall give a rigorous proof.

Remark 2.2. *In his article [5], Littlewood omits many needed points of this proof, we prefer to add some detail to the one contained in [7].*

To prove Theorem 4 we will first need the following lemma:

Lemma 2.1. *If the Riemann Hypothesis is true then*

$$\sum_{n=1}^{\infty} \frac{\mu(s)}{n^s} \tag{11}$$

is convergent and its sum is $\frac{1}{\zeta(s)}$ for every s with $\sigma > \frac{1}{2}$.

Remark 2.3. *Notice the difference with Theorem 3, this time we have $\sigma > \frac{1}{2}$.*

Proof. Begin by showing that:

If is x half an odd integer and $\sigma > -1$ then:

$$\sum_{n<x} \frac{\mu(n)}{n^s} = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{\zeta(s+w)} \frac{x^w}{w} dw + \mathcal{O}\left(\frac{x^2}{T}\right). \tag{12}$$

Remark 2.4. *The truth of the Riemann Hypothesis is not necessary for this equation, it will be required later in the demonstration.*

Fix $n < x$ and consider the integral:

$$\frac{1}{2\pi i} \int_{R^*} \left(\frac{x}{n}\right)^w \frac{dw}{w} = \frac{1}{2\pi i} \left[\int_{-\infty-iT}^{2-iT} \left(\frac{x}{n}\right)^w \frac{dw}{w} + \int_{2-iT}^{2+iT} \left(\frac{x}{n}\right)^w \frac{dw}{w} + \int_{2+iT}^{-\infty+iT} \left(\frac{x}{n}\right)^w \frac{dw}{w} \right]$$

where the contour R^* is the rectangle in the picture below.

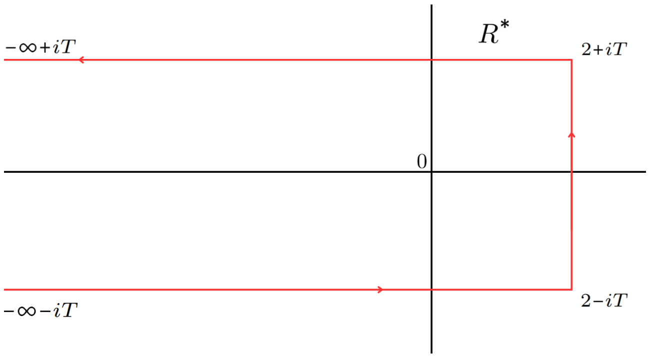

Figure 1: The Rectangle R^*

The only singularity of the function in the rectangle is the one in $w = 0$ where:

$$Res_0\left(\left(\frac{x}{n}\right)^w \frac{1}{w}\right) = \lim_{w\to 0} w\left(\frac{x}{n}\right)^w \frac{1}{w} = \lim_{w\to 0}\left(\frac{x}{n}\right)^w = 1$$

therefore, the Residue Theorem implies:

$$\frac{1}{2\pi i}\left[\int_{-\infty-iT}^{2-iT}\left(\frac{x}{n}\right)^w \frac{dw}{w} + \int_{2-iT}^{2+iT}\left(\frac{x}{n}\right)^w \frac{dw}{w} + \int_{2+iT}^{-\infty+iT}\left(\frac{x}{n}\right)^w \frac{dw}{w}\right] = 1. \qquad (13)$$

Now, integrating by parts, with $f'(w) = \left(\frac{x}{n}\right)^w \Rightarrow f(w) = \frac{\left(\frac{x}{n}\right)^w}{\log\frac{x}{n}}$ and $g(w) = \frac{1}{w} \Rightarrow g'(w) = -\frac{1}{w^2}$, we have:

$$\begin{aligned}\int_{-\infty+iT}^{2+iT}\left(\frac{x}{n}\right)^w \frac{dw}{w} &= \left[\frac{\left(\frac{x}{n}\right)^w}{w\log\frac{x}{n}}\right]_{-\infty+iT}^{2+iT} + \frac{1}{\log\frac{x}{n}}\int_{-\infty+iT}^{2+iT}\left(\frac{x}{n}\right)^w \frac{dw}{w^2} \\ &= \mathcal{O}\left(\frac{\left(\frac{x}{n}\right)^2}{T\log\frac{x}{n}}\right) + \mathcal{O}\left(\frac{\left(\frac{x}{n}\right)^2}{\log\frac{x}{n}}\int_{-\infty}^{+\infty}\frac{du}{u^2+T^2}\right) \qquad (14) \\ &= \mathcal{O}\left(\frac{\left(\frac{x}{n}\right)^2}{T\log\frac{x}{n}}\right).\end{aligned}$$

Using the same argument, we obtain the same result for the integral on the line $(-\infty - iT, 2 - iT)$.

Hence, moving these elements to the right side of equation 13:

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\left(\frac{x}{n}\right)^w \frac{dw}{w} = 1 + \mathcal{O}\left(\frac{\left(\frac{x}{n}\right)^2}{T\log\frac{x}{n}}\right).$$

Multiplying by $\mu(n)n^{-s}$ and summing over all $n < x$ we obtain:

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\sum_{n<x}\frac{\mu(n)}{n^s}\left(\frac{x}{n}\right)^w\frac{dw}{w} = \sum_{n<x}\frac{\mu(n)}{n^s} + \sum_{n<x}\frac{\mu(n)}{n^s}\mathcal{O}\left(\frac{\left(\frac{x}{n}\right)^2}{T\log\frac{x}{n}}\right)$$

$$\Downarrow$$

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{1}{\zeta(s+w)}\frac{x^w}{w}dw = \sum_{n<x}\frac{\mu(n)}{n^s} + \mathcal{O}\left(\frac{x^2}{T}\sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^{\sigma+2}\left|\log\frac{x}{n}\right|}\right) \tag{15}$$

Where we used the fact that $\sum_{n<x}\frac{\mu(n)}{n^{s+w}} = \frac{1}{\zeta(s+w)} + \mathcal{O}(1)$, due to Theorem 3; (we are allowed to use it here since $Re(s+w) = \sigma + 2 > 1$).

Now notice that $\log\frac{x}{n} \neq 0$ because x is half an odd integer, therefore:

$$\left|\log\frac{x}{n}\right| \geq \left|\log\frac{x}{[x]}\right| > 0.$$

Hence:

$$\sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^{\sigma+2}\left|\log\frac{x}{n}\right|} \leq \frac{1}{\left|\log\frac{x}{[x]}\right|}\sum_{n=1}^{\infty}\frac{1}{n^{\sigma+2}} = K\zeta(\sigma+2) \tag{16}$$

so that:

$$\mathcal{O}\left(\frac{x^2}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma+2}\left|\log\frac{x}{n}\right|}\right) = \mathcal{O}\left(\frac{x^2}{T}\right)$$

which proves equation 12.

Remark 2.5. *To complete this demonstration we will use the fact that:*

Theorem 5. *Assuming true the Riemann Hypothesis we have*

$$\zeta(s) = \mathcal{O}(t^{\epsilon}) \tag{17}$$

and

$$\frac{1}{\zeta(s)} = \mathcal{O}(t^{\epsilon}) \tag{18}$$

for every $\sigma > \frac{1}{2}$, ϵ close to zero. (Remember that $t = Im(s)$).

This properties are nontrivial, we added an appendix to this text on our site where you can find a rigorous proof.

To complete the demonstration of this Lemma, assume now that the Riemann Hypothesis is true:

For $0 < \delta < \sigma - \frac{1}{2}$, consider the rectangle R shown in the picture below.

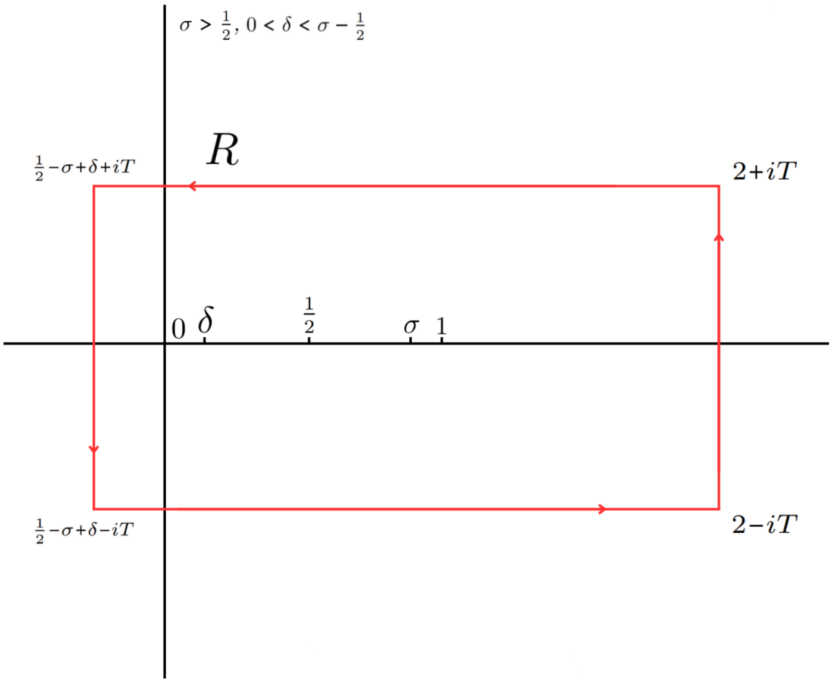

Figure 2: The Rectangle R

We have:

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{1}{\zeta(s+w)}\frac{x^w}{w}dw = \frac{1}{2\pi i}\int_R \frac{1}{\zeta(s+w)}\frac{x^w}{w}dw - \frac{1}{2\pi i}\int_{2+iT}^{\frac{1}{2}-\sigma+\delta+iT}\frac{1}{\zeta(s+w)}\frac{x^w}{w}dw - \frac{1}{2\pi i}\int_{\frac{1}{2}-\sigma+\delta+iT}^{\frac{1}{2}-\sigma+\delta-iT}\frac{1}{\zeta(s+w)}\frac{x^w}{w}dw - \frac{1}{2\pi i}\int_{\frac{1}{2}-\sigma+\delta-iT}^{2-iT}\frac{1}{\zeta(s+w)}\frac{x^w}{w}dw. \tag{19}$$

Compute the first integral using the Residue Theorem; the only singularity in R is the origin because in this integral we have:

$$Re(s+w) = Re(s) + Re(w) \geq \sigma + \frac{1}{2} - \sigma + \delta = \frac{1}{2} + \delta > \frac{1}{2}.$$

Therefore, having assumed true RH, $\frac{1}{\zeta(s+w)}$ doesn't have any singularities inside the contour.

In 0 the residue is:

$$Res_0\left(\frac{1}{\zeta(s+w)}\frac{x^w}{w}\right) = \lim_{w\to 0}\frac{w}{\zeta(s+w)}\frac{x^w}{w} = \lim_{w\to 0}\frac{x^w}{\zeta(s+w)} = \frac{1}{\zeta(s)}.$$

Hence equation 19 becomes:

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{1}{\zeta(s+w)}\frac{x^w}{w}dw = \frac{1}{\zeta(s)} + \frac{1}{2\pi i}\int_{2-iT}^{\frac{1}{2}-\sigma+\delta-iT}\frac{1}{\zeta(s+w)}\frac{x^w}{w}dw + \frac{1}{2\pi i}\int_{\frac{1}{2}-\sigma+\delta-iT}^{\frac{1}{2}-\sigma+\delta+iT}\frac{1}{\zeta(s+w)}\frac{x^w}{w}dw + \frac{1}{2\pi i}\int_{\frac{1}{2}-\sigma+\delta+iT}^{2+iT}\frac{1}{\zeta(s+w)}\frac{x^w}{w}dw. \tag{20}$$

Therefore, using equation 12:

$$\sum_{n<x}\frac{\mu(n)}{n^s} + \mathcal{O}\left(\frac{x^2}{T}\right) = \frac{1}{\zeta(s)} + \frac{1}{2\pi i}\int_{2-iT}^{\frac{1}{2}-\sigma+\delta-iT}\frac{1}{\zeta(s+w)}\frac{x^w}{w}dw + \frac{1}{2\pi i}\int_{\frac{1}{2}-\sigma+\delta-iT}^{\frac{1}{2}-\sigma+\delta+iT}\frac{1}{\zeta(s+w)}\frac{x^w}{w}dw + \frac{1}{2\pi i}\int_{\frac{1}{2}-\sigma+\delta+iT}^{2+iT}\frac{1}{\zeta(s+w)}\frac{x^w}{w}dw. \tag{21}$$

The first and third integral are:

$$\begin{aligned}\int_{2-iT}^{\frac{1}{2}-\sigma+\delta-iT}\frac{1}{\zeta(s+w)}\frac{x^w}{w}dw &= \int_{2}^{\frac{1}{2}-\sigma+\delta}\frac{1}{\zeta(s+u-iT)}\frac{x^{u-iT}}{u-iT}du \\ &= \mathcal{O}\left(T^{\epsilon-1}\int_{2}^{\frac{1}{2}-\sigma+\delta}x^u du\right) = \mathcal{O}\left(T^{\epsilon-1}x^2\right)\end{aligned} \tag{22}$$

where we used Theorem 5.

While using 5 for the second integral yields:

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\frac{1}{2}-\sigma+\delta-iT}^{\frac{1}{2}-\sigma+\delta+iT}\frac{1}{\zeta(s+w)}\frac{x^w}{w}dw = \int_{-T}^{T}\frac{1}{\zeta(s+\frac{1}{2}-\sigma+\delta+it)}\frac{x^{\frac{1}{2}-\sigma+\delta+it}}{\frac{1}{2}-\sigma+\delta+it}idt \\ &= \mathcal{O}\left(x^{\frac{1}{2}-\sigma+\delta}\int_{-T}^{T}(1+|t|)^{\epsilon-1}dt\right) = \mathcal{O}\left(x^{\frac{1}{2}-\sigma+\delta}T^{\epsilon}\right).\end{aligned} \tag{23}$$

Hence:

$$\sum_{n<x}\frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} + \mathcal{O}\left(T^{\epsilon-1}x^2\right) + \mathcal{O}\left(x^{\frac{1}{2}-\sigma+\delta}T^{\epsilon}\right).$$

Considering for example $T = x^3$, the $\mathcal{O}$-terms tend to zero as $x \to \infty$ and the result follows. □

Let's see how we can use this result to prove Theorem 4:

Proof. of Theorem 4:

Remember the definition of the Mertens function:

$$M(x) := \sum_{n\leq x}\mu(n).$$

Choose x half an odd number and use equation 12 with $s = 0$ to obtain:

$$\sum_{n<x} \frac{\mu(n)}{n^0} = M(x) = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{\zeta(w)} \frac{x^w}{w} + \mathcal{O}\left(\frac{x^2}{T}\right). \tag{24}$$

Using the same argument used for the last proof but with a different rectangle R' shown in the picture below, we have:

$$\frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{\zeta(w)} \frac{x^w}{w} = \frac{1}{2\pi i} \int_{2-iT}^{\frac{1}{2}+\delta-iT} \frac{1}{\zeta(w)} \frac{x^w}{w} dw + \frac{1}{2\pi i} \int_{\frac{1}{2}+\delta-iT}^{\frac{1}{2}+\delta+iT} \frac{1}{\zeta(w)} \frac{x^w}{w} dw + \frac{1}{2\pi i} \int_{\frac{1}{2}+\delta+iT}^{2+iT} \frac{1}{\zeta(w)} \frac{x^w}{w} dw + \mathcal{O}\left(\frac{x^2}{T}\right). \tag{25}$$

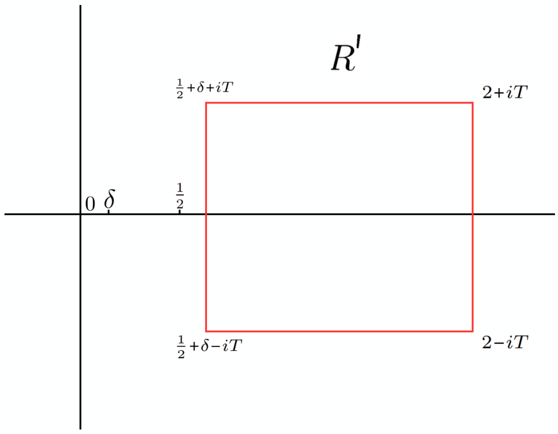

Figure 3: The Rectangle R', notice that it does not contain any singularities.

Therefore, estimating like in equation 23 we have:

$$M(x) = \mathcal{O}\left(T^{\epsilon} x^{\frac{1}{2}+\delta}\right) + \mathcal{O}\left(T^{\epsilon-1} x^2\right). \tag{26}$$

Taking $T = x^2$, we have:

$$M(x) = \mathcal{O}\left(x^{2\epsilon+\frac{1}{2}+\delta}\right) + \mathcal{O}\left(x^{\epsilon}\right) = \mathcal{O}\left(x^{\frac{1}{2}+\epsilon}\right)$$

the Theorem follows for x any half an odd integer and so generally. □

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Thank you!

We hope this lesson has been beneficial in studying this interesting topic.
For more lessons or demonstrations, visit our website.

References

[1] Tom M Apostol. *Introduction to analytic number theory.* Springer Science & Business Media, 2013.

[2] Gregory Convertito and David Cruz-Uribe. *The Stieltjes Integral.* Chapman and Hall/CRC, 2023.

[3] Harold M Edwards. *RiemannÆs zeta function.* Vol. 58. Academic press, 1974.

[4] GH Hardy. *An introduction to the theory of numbers.* Oxford Science Publication, 1979.

[5] JE Littlewood. "Quelques conséquences de l'hypothese que la fonction ζ (s) de Riemann n'a pas de zéros dans le demi-plan Re(s)> 1/2". In: *CRAS Paris* 154 (1912), pp. 263–266.

[6] August Ferdinand Möbius. "Über eine besondere Art von Umkehrung der Reihen." In: (1832).

[7] Edward Charles Titchmarsh and David Rodney Heath-Brown. *The theory of the Riemann zeta-function.* Oxford university press, 1986.