The cases in which the Riemann Zeta can be evaluated explicitly are not common; however, these are also not impossible.

It's somewhat easy to prove that the function equals zero for negative even integers smaller than zero; these are usually called **Trivial Zeros** (see our lesson regarding Riemann's original paper).

Another commonly known fact is that $\zeta(0) = -\frac{1}{2}$. on the other hand, not everyone knows that this is a particular case of the connection between the Zeta Function and the Bernoulli Numbers.

Remark 1. *Most of the following lesson is inspired by [1].*

Definition 1. *For any $x \in \mathbb{C}$, the **Bernoulli polynomial** $B_n(x)$ is defined by the equation:*

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n, \qquad \text{where } |t| < 2\pi. \tag{1}$$

*The numbers $B_n(0)$ are called **Bernoulli numbers** and are denoted by B_n.*

Theorem 1. *The functions $B_n(x)$ are polynomials in x given by:*

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}. \tag{2}$$

Proof. Notice first that, by definition:

$$\sum_{n=0}^{\infty} \frac{B_n(0)}{n!} t^n = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n = \frac{t}{e^t - 1}.$$

Therefore:

$$\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n = \frac{t}{e^t - 1} \cdot e^{xt} = \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} t^n \right) \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} t^n \right)$$

where we used the series definition of the exponential function.

We can compute the product of the two series to obtain:

$$\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n = \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} t^n \right) \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} t^n \right) = \sum_{n=0}^{\infty} t^n \left(\sum_{k=0}^{n} \frac{B_k}{k!} \frac{x^{n-k}}{(n-k)!} \right)$$

$$\Downarrow$$

$$\frac{B_n(x)}{n!} = \sum_{k=0}^{n} \frac{B_k}{k!} \frac{x^{n-k}}{(n-k)!}$$

$$\Downarrow$$

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}$$

and the Theorem follows. □

Theorem 2. *For every integer $n \geq 0$ we have:*

$$\zeta(-n) = -\frac{B_{n+1}(1)}{n+1}. \tag{3}$$

This is a corollary of the following Theorem:

Theorem 3. *The Riemann Zeta function satisfies the equation:*

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{-\infty}^{(0,+)} e^t \frac{t^{s-1}}{1-e^t} \mathrm{d}t \tag{4}$$

Here the integration contour is C, a loop around the negative real axis; it starts at $-\infty$, encircles the origin once in the positive direction without enclosing any of the points $t = \pm 2\pi i, \pm 4\pi i, \cdots$, and returns to $-\infty$.
t^{-s} has its principal value where t crosses the positive real axis and is continuous.

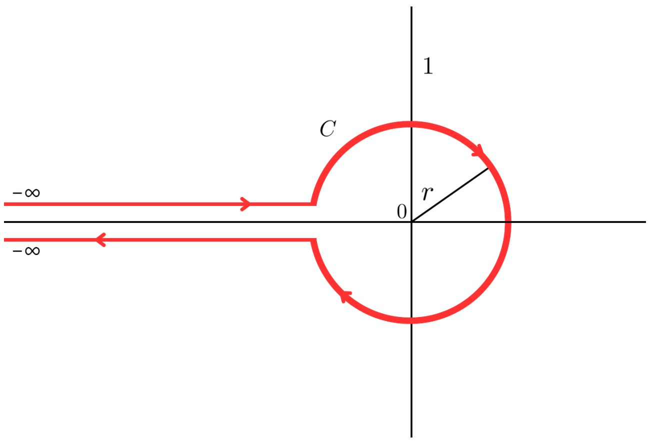

Figure 1: The Contour C

You can find a rigorous proof here.

Proof of Theorem 2. Using Theorem 3, we have:

$$\zeta(-n) = \frac{\Gamma(1+n)}{2\pi i}\int_{-\infty}^{(0,+)} e^t \frac{t^{-n-1}}{1-e^t}\mathrm{d}t = \frac{n!}{2\pi i}\int_{-\infty}^{(0,+)} e^t \frac{t^{-n-1}}{1-e^t}\mathrm{d}t = n!\left(\mathrm{Res}_{t=0}\left(e^t\frac{t^{-n-1}}{1-e^t}\right)\right).$$

where we used definition of the Gamma Function and the Residue Theorem.

Now, with some simple algebra:

$$\begin{aligned}\mathrm{Res}_{t=0}\left(e^t\frac{t^{-n-1}}{1-e^t}\right) &= -\mathrm{Res}_{t=0}\left(e^t\frac{t^{-n-1}}{e^t-1}\right) = -\mathrm{Res}_{t=0}\left(t^{-n-2}\frac{te^t}{e^t-1}\right)\\ &= -\mathrm{Res}_{t=0}\left(t^{-n-2}\sum_{m=0}^{\infty}\frac{B_m(1)}{m!}t^m\right) = -\frac{B_{n+1}(1)}{(n+1)!}.\end{aligned} \tag{5}$$

Therefore

$$\zeta(-n) = n!\left(\mathrm{Res}_{t=0}\left(e^t\frac{t^{-n-1}}{1-e^t}\right)\right) = -n!\frac{B_{n+1}(1)}{(n+1)!} = -\frac{B_{n+1}(1)}{n+1}.$$

□

Theorem 4. *The Bernoulli polynomials $B_n(x)$ satisfy the equation:*

$$B_n(x+1) - B_n(x) = nx^{n-1}, \qquad \textit{if } n \geq 1.$$

Therefore

$$B_n(0) = B_n(1), \qquad \textit{if } n \geq 2.$$

Remark 2. *With this result, equation 3 can be written as:*

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}.$$

Proof. It is simple to see that:

$$t\frac{e^{(x+1)t}}{e^t-1} - t\frac{e^{xt}}{e^t-1} = te^{xt}$$

this can be written, by definition of the Bernoulli polynomials, as:

$$\sum_{n=0}^{\infty}\frac{B_n(x+1)-B_n(x)}{n!}t^n = \sum_{n=0}^{\infty}\frac{x^n}{n!}t^{n+1}$$

$$\Downarrow$$

$$\frac{B_n(x+1)-B_n(x)}{n!} = \frac{x^{n-1}}{(n-1)!}$$

$$\Downarrow$$

$$B_n(x+1) - B_n(x) = nx^{n-1}.$$

Evaluating for $x = 0$ we complete the proof. □

Theorem 5. *If k is a positive integer, we have:*

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k} B_{2k}}{2(2k)!}. \tag{6}$$

Proof. Remember the Second Functional equation for the Zeta Function:

$$\zeta(1-s) = 2(2\pi)^{-s} \cos\left(\frac{\pi}{2}s\right) \Gamma(s)\zeta(s)$$

this implies:

$$\zeta(1-2k) = 2(2\pi)^{-2k} \cos(\pi k)\, \Gamma(2k)\zeta(2k).$$

The left side of the equation is exactly $\zeta(-(2k-1))$ and we can therefore apply Theorem 2 to obtain:

$$-\frac{B_{2k}}{2k} = 2(2\pi)^{-2k}(2k-1)!(-1)^k \zeta(2k) \tag{7}$$

$$\Downarrow$$

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k} B_{2k}}{2(2k)!}.$$

□

POSITIVE
INCREMENT

Thank you!

We hope this lesson has been beneficial in studying this interesting topic.
For more lessons or demonstrations, visit our website.

References

[1] Tom M Apostol. *Introduction to analytic number theory*. Springer Science & Business Media, 2013.