In this lesson, we discuss a fascinating formulation of the Riemann Hypothesis connected to a property of natural numbers. While this property may be simple to understand, its demonstration has eluded more than a century of research.

We start by introducing the so-called Liouville function and explain its connection to the Riemann Zeta Function.

Liouville's function has long been studied in connection with the Riemann Hypothesis. Insights regarding this topic can be found in many books; two excellent examples are [1] and [2]. This lesson is primarily inspired by these two works, as well as the article "The Liouville Function and the Riemann Hypothesis" by MJ Mossinghoff and TS Trudgian [6].

Let's start with the definition:

Definition 1. *For any positive integer* n*, with prime factorization* $n = p_1^{a_1} p_1^{a_2} \cdots p_k^{a_k}$*, call* $\Omega(n) = a_1 + a_2 + \cdots a_k$ *the number of primes in the factorization counted with multiplicity.*

The ***Liouville Function*** $\lambda(n)$ *is defined as:*

$$\lambda(n) := \begin{cases} 1 & \text{if } n = 1; \\ (-1)^{\Omega(n)} & \text{if } n \neq 1. \end{cases} \tag{1}$$

1 Connection to the Riemann Hypothesis

To understand the connection to the Riemann Hypothesis we must first link this function to the Riemann Zeta. To do this, consider the fraction:

$$\frac{\zeta(2s)}{\zeta(s)}$$

for $s \neq \frac{1}{2}$. Remember that, for $Re(s) > 0$ the roots of the Zeta Function are contained in the **Critical Strip** $0 < Re(s) < 1$, this, added to the fact that those are symmetric about the straight line $Re(s) = \frac{1}{2}$ imply that, in the right-hand side of the complex plane:

$$\zeta(s) = 0 \Rightarrow \zeta(2s) \neq 0.$$

Therefore s with $Re(s) > 0$ is a singularity of $\frac{\zeta(2s)}{\zeta(s)}$ if and only if s is a **Non Trivial Zero of the Riemann Zeta Function.**

Remember the Euler product for the Riemann Zeta Function:

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \tag{2}$$

valid for $Re(s) > 1$.

Using this equation we can link Liouville's Function to the Riemann Zeta:

$$\begin{aligned} \frac{\zeta(2s)}{\zeta(s)} &= \frac{\prod_p (1-p^{-2s})^{-1}}{\prod_p (1-p^{-s})^{-1}} = \prod_p \frac{(1-p^{-s})}{(1-p^{-2s})} = \prod_p \frac{(1-p^{-s})}{(1-p^{-s})(1+p^{-s})} \\ &= \prod_p \frac{1}{(1+p^{-s})} = \prod_p \sum_{k=0}^{\infty} \frac{(-1)^k}{p^{ks}} \end{aligned} \tag{3}$$

where we used the fact that $Re(s) > 1 \Rightarrow |p^{-s}| < 1$, therefore we can write the last term as an alternating geometric series: $\frac{1}{1+r} = \sum_{k=0}^{\infty} (-1)^k r^k$.

Let's think about what the product $\prod_p \sum_{k=0}^{\infty} \frac{(-1)^k}{p^{ks}}$ yields:

$$\prod_p \sum_{k=0}^{\infty} \frac{(-1)^k}{p^{ks}} = \left(1 - \frac{1}{2^s} + \frac{1}{2^{2s}} - \cdots\right)\left(1 - \frac{1}{3^s} + \frac{1}{3^{2s}} - \cdots\right)\cdots$$

In the denominators we find every possible combination of products of primes of any power, that is to say, one denominator for every integer $n \geq 1$:

While the numerator is the product (-1) repeated k times where $k = \Omega(n)$, thus:

$$\frac{\zeta(2s)}{\zeta(s)} = \prod_p \sum_{k=0}^{\infty} \frac{(-1)^k}{p^{ks}} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}. \tag{4}$$

Computing a weighted sum like $\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}$ is a method often used in Analytic Number Theory (see [1] chapter 2 and 3).

We will see that this is connected to a particular case of the function $L_\alpha(x)$ defined as:

$$L_\alpha(x) := \sum_{n \leq x} \frac{\lambda(n)}{n^\alpha}.$$

The instances $\alpha = 0$ and $\alpha = 1$ have been studied thoroughly, often in connection to the Riemann Zeta Function.

To understand why it is the case, let's work more on equation 4, writing $\lambda(n) = L_0(n) - L_0(n-1)$:

$$\begin{aligned}\frac{\zeta(2s)}{\zeta(s)} &= \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \sum_{n=1}^{\infty} \frac{L_0(n) - L_0(n-1)}{n^s} = \sum_{n=1}^{\infty} \frac{L_0(n)}{n^s} - \sum_{n=1}^{\infty} \frac{L_0(n-1)}{n^s} \\ &= \sum_{n=1}^{\infty} \frac{L_0(n)}{n^s} - \sum_{n=0}^{\infty} \frac{L_0(n)}{(n+1)^s} = \sum_{n=1}^{\infty} L_0(n) \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right)\end{aligned} \tag{5}$$

where we used the fact that by definition $L_0(0) = 0$.

The last term can be written in integral form using the fact that:

$$\left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) = \int_n^{n+1} \frac{s}{x^{s+1}} dx.$$

Therefore:

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} L_0(n) \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) = \sum_{n=1}^{\infty} L_0(n) \int_n^{n+1} \frac{s}{x^{s+1}} dx = s \int_1^{\infty} \frac{L_0(x)}{x^{s+1}} dx. \tag{6}$$

This last equality brings us to a few important conclusions:

First, this expression of the function $\frac{\zeta(2s)}{\zeta(s)}$ is not only valid for $Re(s) > 1$ but its domain of definition depends on the convergence of the integral $s \int_1^{\infty} \frac{L_0(x)}{x^{s+1}} dx$.

Therefore, if one could prove that:

$$L_0(x) = \mathcal{O}(\sqrt{x}) \tag{7}$$

then $\frac{\zeta(2s)}{\zeta(s)}$ would converge for $Re(s) > \frac{1}{2}$ and the roots of the Zeta Function would be constrained in the Critical Line, implying the Riemann Hypothesis!

This leads us to a very interesting reformulation:

It is a known result in probability, that for a *Simple Random Walk on* $\mathbb{Z}$, i.e. a random sequence of 1s and -1s with equal probability, the expected value of the absolute value of the sum is of the order of $\sqrt{n}$.

Therefore, proving that the sum $L_0(x)$ behaves as a random walk would imply 7, that is to say:

The Riemann Hypothesis would be implied by proving that every integer has equal probability of having an odd number or an even number of distinct prime factors.

2 Developments and conclusions

To this interesting conclusion, we add some good news and some bad news:

On the bright side, Landau's Theorem regarding singularities of the Laplace transform of non-negative functions (see for example [5]) implies that we could prove less than $L_0(x) = \mathcal{O}(\sqrt{x})$. In particular, the Riemann Hypothesis would follow if any of the following inequalities holds for sufficiently large x:

$$\frac{L_0(x)}{\sqrt{x}} < C, \qquad \frac{L_0(x)}{\sqrt{x}} > -C, \qquad L_1(x)\sqrt{x} < C, \qquad L_1(x)\sqrt{x} > -C.$$

Meanwhile, on the downside, Ingham proved in 1942 that bounding these functions would imply much more than just the Riemann Hypothesis; it would follow that there are infinitely many integer relations among the ordinates of the zeros of $\zeta(s)$ on the upper half-plane of the critical line [4].

Although intriguing, the existence of such relationships is not supported by any evidence, and there appears to be no valid reason to believe they are true. The best approach to the problem seems to be to focus on proving 7.

For those interested in this approach to the Riemann Hypothesis, we recommend the paper "The Liouville Function and the Riemann Hypothesis" by Michael J. Mossinghoff and Timothy S. Trudgian. This paper provides bounds on the oscillation of the function $L_0(x)$. Additionally, the article "The Final and Exhaustive Proof of the Riemann Hypothesis from First Principles" by K. Eswaran attempts to demonstrate the Hypothesis by exploring its connection to the Random Walk [3].

POSITIVE INCREMENT

Thank you!

We hope this lesson has been beneficial in studying this interesting topic.
For more lessons or demonstrations, visit our website.

References

[1] Tom M Apostol. *Introduction to analytic number theory*. Springer Science & Business Media, 2013.

[2] Peter Borwein et al. *The Riemann hypothesis: a resource for the afficionado and virtuoso alike*. Springer, 2008.

[3] K Eswaran. "The Final and Exhaustive Proof of the Riemann Hypothesis from First Principles". In: *Hyperlink: Final* (2018).

[4] Albert E Ingham. "On two conjectures in the theory of numbers". In: *American journal of mathematics* 64.1 (1942), pp. 313–319.

[5] Hugh L Montgomery and Robert C Vaughan. *Multiplicative number theory I: Classical theory*. 97. Cambridge university press, 2007.

[6] Michael J Mossinghoff and Timothy S Trudgian. "The Liouville function and the Riemann hypothesis". In: *Exploring the Riemann Zeta Function: 190 years from Riemann's Birth* (2017), pp. 201–221.