The contributions of the legendary mathematician Srinivasa Ramanujan are impossible to quantify. Despite his short life, he compiled nearly 3900 results, many completely novel.
The Ramanujan's sum appeared in his 1918 paper "On certain trigonometrical sums and their applications in the theory of numbers," where he writes:
"These sums are obviously of great interest, and a few of their properties have been discussed already. But, so far as I know, they have never been considered from the point of view which I adopt in this paper; and I believe that all the results which it contains are new".

The following series of Theorems was initially proved by Ramanujan; we give a more detailed version of the demonstrations from [2].

Definition 1. *For any integer $k \geq 1$, define the* ***Ramanujan's Sum*** $c_k(n)$ *as:*

$$c_k(n) := \sum_{\substack{h<k \\ (h,k)=1}} e^{-\frac{2nh\pi i}{k}} = \sum_{\substack{h<k \\ (h,k)=1}} \cos\left(\frac{2nh\pi}{k}\right). \tag{1}$$

Theorem 1. *For all integers $k \geq 1$, we have:*

$$c_k(n) = \sum_{\substack{d|k \\ d|n}} \mu\left(\frac{k}{d}\right) d. \tag{2}$$

Proof. Start by considering the sum:

$$\eta_k(n) := \sum_{m=0}^{k-1} e^{-\frac{2nm\pi i}{k}}$$

if $k|n$, the exponent of every term is a multiple of 2π, therefore we are summing 1 for k times, which implies:

$$\eta_k(n) = \sum_{m=0}^{k-1} e^{-\frac{2nm\pi i}{k}} = k \quad \text{if } k|n. \tag{3}$$

While if $k \nmid n$ then the sum is geometric and $e^{-\frac{2n\pi i}{k}} \neq 1$, therefore:

$$\eta_k(n) = \sum_{m=0}^{k-1} e^{-\frac{2nm\pi i}{k}} = \sum_{m=0}^{k-1} \left(e^{-\frac{2n\pi i}{k}}\right)^m = \frac{1-\left(e^{-\frac{2n\pi i}{k}}\right)^k}{1-e^{-\frac{2n\pi i}{k}}} = \frac{1-e^{-2n\pi i}}{1-e^{-\frac{2n\pi i}{k}}} = 0. \quad (4)$$

Now, by definition:

$$\sum_{d|k} c_d(n) = \sum_{d|k} \sum_{\substack{(r,d)=1 \\ r<d}} e^{-\frac{2nr\pi i}{d}} = \eta_k(n). \quad (5)$$

Hence, using Möbius inversion formula and equation 3 we have:

$$c_k(n) = \sum_{d|k} \mu\left(\frac{k}{d}\right)\eta_d(n) = \sum_{\substack{d|k \\ d|n}} \mu\left(\frac{k}{d}\right) d.$$

□

Corollary 1. *For all complex numbers s with $Re(s) > 1$ we have:*

$$\sum_{k=1}^{\infty} \frac{c_k(n)}{k^s} = \frac{\sigma_{1-s}(n)}{\zeta(s)} \quad (6)$$

and

$$\sum_{n=1}^{\infty} \frac{c_k(n)}{n^s} = \zeta(s) \sum_{d|k} \mu\left(\frac{k}{d}\right) d^{1-s}. \quad (7)$$

where $\sigma_a(n)$ denotes the sum of the divisors of n to the a-th power, that is to say:

$$\sigma_a(n) := \sum_{k|n} k^a.$$

Remark 1. *By definition $\sigma_0(n)$ is the number of divisors of n, we will denote this function $d(n)$.*

Proof. The result of Theorem 1 can be written, calling $r = \frac{k}{d}$ as:

$$c_k(n) = \sum_{\substack{dr=k \\ d|n}} \mu(r) d.$$

Therefore:

$$c_k(n) \cdot \frac{1}{k^s} = \sum_{\substack{dr=k \\ d|n}} \mu(r) d \cdot \frac{1}{k^s}$$

$$\Downarrow$$

$$\frac{c_k(n)}{k^s} = \sum_{\substack{dr=k \\ d|n}} \mu(r) d \cdot \frac{1}{(dr)^s} = \sum_{\substack{dr=k \\ d|n}} \frac{\mu(r)}{r^s} d^{1-s}.$$

Hence:

$$\sum_{k=1}^{\infty} \frac{c_k(n)}{k^s} = \sum_{\substack{r \\ d|n}} \frac{\mu(r)}{r^s} d^{1-s} = \frac{\sigma_{1-s}(n)}{\zeta(s)}.$$

Where we used the known Connection to the Möbius Function. This proves equation 6.

We also have, using again Theorem 1:

$$\sum_{n=1}^{\infty}\frac{c_k(n)}{n^s}=\sum_{n=1}^{\infty}\frac{1}{n^s}\left(\sum_{\substack{d|k\\d|n}}\mu\left(\frac{k}{d}\right)d\right)=\left(\sum_{d|k}\mu\left(\frac{k}{d}\right)d\right)\sum_{m=1}^{\infty}\frac{1}{(md)^s}=\left(\sum_{d|k}\mu\left(\frac{k}{d}\right)d\right)\frac{1}{d^s}\sum_{m=1}^{\infty}\frac{1}{m^s}$$
$$=\zeta(s)\sum_{d|k}\mu\left(\frac{k}{d}\right)d^{1-s} \tag{8}$$

which is exactly equation 7. □

Corollary 2. *For all complex numbers s with $Re(s)>1$ we have:*

$$\sum_{n=1}^{\infty}\frac{c_k(qn)}{n^s}=\zeta(s)\sum_{\delta|k}\delta^{1-s}\mu\left(\frac{k}{\delta}\right)(q,\delta)^s \tag{9}$$

Remark 2. *(q,δ) indicates the greatest common divisor of q and δ.*

Proof. Using again Theorem 1 we have:

$$\sum_{n=1}^{\infty}\frac{c_k(qn)}{n^s}=\sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{\substack{\delta|k\\\delta|qn}}\mu\left(\frac{k}{\delta}\right)\delta$$

For a given δ, $n=\frac{\delta}{q}j$ for some integer j, therefore, n runs through the multiples of $\frac{\delta}{q}$ which are integers. If $\frac{\delta}{q}$ in its lowest terms is $\frac{\delta_1}{q_1}$, these are the numbers $\delta_1, 2\delta_1\cdots$.

This implies:

$$\sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{\substack{\delta|k\\\delta|qn}}\mu\left(\frac{k}{\delta}\right)\delta=\sum_{\delta|k}\delta\mu\left(\frac{k}{\delta}\right)\sum_{r=1}^{\infty}\frac{1}{(r\delta_1)^s}=\sum_{\delta|k}\delta\mu\left(\frac{k}{\delta}\right)\frac{1}{\delta_1^s}\sum_{r=1}^{\infty}\frac{1}{r^s}=\zeta(s)\sum_{\delta|k}\delta\mu\left(\frac{k}{\delta}\right)\delta_1^{-s}.$$

Since $\delta_1=\frac{\delta}{(q,\delta)}$ by definition, we conclude:

$$\sum_{n=1}^{\infty}\frac{c_k(qn)}{n^s}=\zeta(s)\sum_{\delta|k}\delta\mu\left(\frac{k}{\delta}\right)\left(\frac{\delta}{(q,\delta)}\right)^{-s}=\zeta(s)\sum_{\delta|k}\delta^{1-s}\mu\left(\frac{k}{\delta}\right)(q,\delta)^s.$$

□

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Thank you!

We hope this lesson has been beneficial in studying this interesting topic.
For more lessons or demonstrations, visit our website.

References

[1] Marcel Riesz, Garding Lars, and Lars Hörmander. "Sur L'hypothese de Riemann". In: *Collected Papers*. Springer, 1988, pp. 165–170.

[2] Edward Charles Titchmarsh and David Rodney Heath-Brown. *The theory of the Riemann zeta-function*. Oxford university press, 1986.