Theorem 1 (Ramanujan's Master Theorem)**.** *Let $\phi(s)$ be an analytic complex function, defined on the half-plane*

$$H(\delta) = \{s \in \mathbb{C} : Re(s) \geq -\delta\}$$

for some $0 < \delta < 1$. Suppose also that, for some $A < \pi$, ϕ satisfies the growth condition:

$$|\phi(\sigma + ib)| < Ce^{P\sigma + A|b|}$$

for all $s = \sigma + ib \in H(\delta)$.

Then

$$\int_0^\infty t^{s-1} \sum_{n=0}^{\infty} \phi(n) \frac{(-t)^n}{n!} = \Gamma(s)\phi(-s) \tag{1}$$

for all $0 < Re(s) < \delta$.

Remark 1. *The condition $0 < Re(s) < \delta$ is necessary to ensure the convergence of the integral without imposing any additional hypothesis on the function $\phi(s)$. It is evident that if the integral converges in a larger region of the complex plane, the result remains valid due to the principle of analytic continuation.*

Ramanujan's Master Theorem is a corollary of another formula, also first found in Ramanujan's notebooks:

Theorem 2. *Under the hypothesis of Theorem 1 we have:*

$$\int_0^\infty t^{s-1} \sum_{n=0}^{\infty} \phi(n)(-t)^n dt = \frac{\pi}{\sin(s\pi)} \phi(-s). \tag{2}$$

The following demonstration takes inspiration by Hardy's original proof [2] and the more recent semi-expository paper [1] that also discusses a multi-dimensional extension of the theorem.

Proof. Let $1 < t < e^{-P}$, the hypothesis on the growth of the function $\phi(s)$ ensures that the series:

$$\Phi(t) := \sum_{n=0}^{\infty} \phi(n)(-t)^n$$

converges (this can be simply seen using the root test).

Consider the integral on the contour C seen in Figure 1:

$$\frac{1}{2\pi i}\int_C \frac{\pi}{\sin(\pi s)}\phi(-s)t^{-s}ds = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\pi}{\sin(\pi s)}\phi(-s)t^{-s}ds + \frac{1}{2\pi i}\int_{S_T} \frac{\pi}{\sin(\pi s)}\phi(-s)t^{-s}ds$$

with $-\frac{2}{3} \le c - T \le -\frac{1}{2}$. This hypothesis on T is only to ensure that, while traveling on this contour, the function $\frac{\pi}{\sin(\pi s)}$ avoids its singularity in $s = -1$.

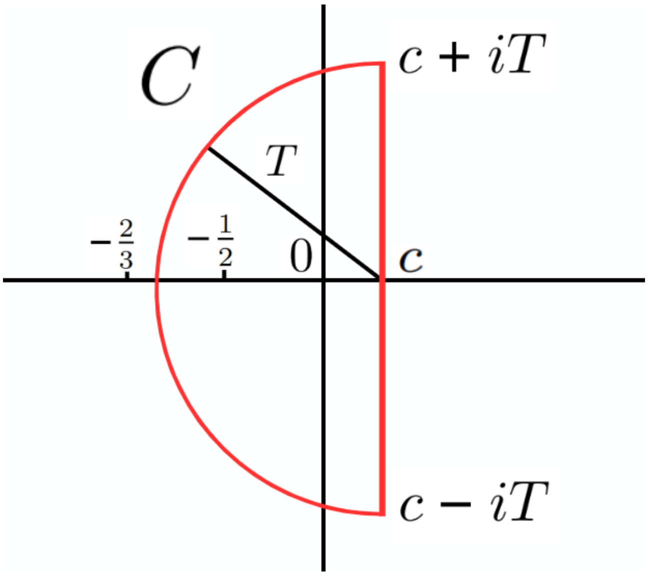

Figure 1: The contour C

Remark 2. *Mind that $c < \delta$ and $\phi(s)$ is well defined for $Re(s) \ge -\delta$. On the contour C we have $Re(s) \le c \Rightarrow Re(-s) \ge -c > -\delta \Rightarrow \phi(-s)$ is well defined. This stays true in the collection of contours that will be later defined.*

Focus on the integral on the semi circumference S_T, calling $s = c + Te^{i\theta}$ we have:

$$\begin{aligned}
&\left|\int_{S_T} \frac{\pi}{\sin(\pi s)}\phi(-s)t^{-s}ds\right| \le \int_{S_T}\left|\frac{\pi}{\sin(\pi s)}\phi(-s)t^{-s}\right| ds \\
&= \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\left|\frac{\pi}{\sin(\pi(c+Te^{i\theta}))}\phi(-c-Te^{i\theta})t^{-c-Te^{i\theta}} iTe^{i\theta}\right| d\theta \\
&= T\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\left|\frac{\pi}{\sin(\pi(c+Te^{i\theta}))}\right|\left|\phi(-c-Te^{i\theta})\right|\left|t^{-c-Te^{i\theta}}\right| d\theta.
\end{aligned} \tag{3}$$

By hypothesis we have:

$$\phi(-c - Te^{i\theta}) = \phi(-c - T\cos(\theta) - iT\sin(\theta))$$
$$\Downarrow$$
$$\left|\phi(-c - Te^{i\theta})\right| < Ce^{P(-c-T\cos(\theta))+AT|\sin(\theta)|}$$

and $1 < t < e^{-P}$, therefore:

$$\begin{aligned}
&\left|\int_{S_T} \frac{\pi}{\sin(\pi s)}\phi(-s)t^{-s}ds\right| = T\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \left|\frac{\pi}{\sin(\pi(c+Te^{i\theta}))}\right| \left|\phi(-c - Te^{i\theta})\right| \left|t^{-c-Te^{i\theta}}\right| d\theta \\
&< CT\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \left|\frac{\pi}{\sin(\pi(c+Te^{i\theta}))}\right| e^{-Pc-PT\cos(\theta)+AT|\sin(\theta)|} \left|e^{Pc+PT\cos(\theta)+iPT\sin(\theta)}\right| d\theta \\
&= CT\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \left|\frac{\pi}{\sin(\pi(c+Te^{i\theta}))}\right| e^{AT|\sin(\theta)|}d\theta.
\end{aligned} \tag{4}$$

Focus now on the term $\frac{\pi}{|\sin(\pi(c+Te^{i\theta}))|}$, start by using Euler's identity to obtain:

$$\left|\sin(\pi(c+Te^{i\theta}))\right| = |\sin(\pi c + \pi T\cos(\theta) + i\pi T\sin(\theta))|$$

then, remembering that $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$ and that $\sin(ix) = -i\sinh(-x)$, $\cos(ix) = \cosh(-x)$ we find:

$$\begin{aligned}
&\left|\sin(\pi(c+Te^{i\theta}))\right| = |\sin(\pi c + \pi T\cos(\theta) + i\pi T\sin(\theta))| \\
&= |\sin(\pi c + \pi T\cos(\theta))\cos(i\pi T\sin(\theta)) + \cos(\pi c + \pi T\cos(\theta))\sin(i\pi T\sin(\theta))| \\
&= |\sin(\pi c + \pi T\cos(\theta))\cosh(-\pi T\sin(\theta)) + \cos(\pi c + \pi T\cos(\theta))(-i\sinh(-\pi T\sin(\theta)))| \\
&= |\sin(\pi c + \pi T\cos(\theta))\cosh(\pi T\sin(\theta)) + i\cos(\pi c + \pi T\cos(\theta))\sinh(\pi T\sin(\theta))|.
\end{aligned} \tag{5}$$

This final equality is already separated into its real and imaginary parts, allowing for a straightforward calculation of the absolute value:

$$\begin{aligned}
&|\sin(\pi c + \pi T\cos(\theta))\cosh(\pi T\sin(\theta)) + i\cos(\pi c + \pi T\cos(\theta))\sinh(\pi T\sin(\theta))| \\
&= \sqrt{\sin^2(\pi c + \pi T\cos(\theta))\cosh^2(\pi T\sin(\theta)) + \cos^2(\pi c + \pi T\cos(\theta))\sinh^2(\pi T\sin(\theta))}.
\end{aligned} \tag{6}$$

Remember now that:

$$\cosh^2(\pi T\sin(\theta)) = 1 + \sinh^2(\pi T\sin(\theta))$$

and

$$\cos^2(\pi c + \pi T\cos(\theta)) = 1 - \sin^2(\pi c + \pi T\cos(\theta)).$$

Hence:

$$\begin{aligned}
&\sqrt{\sin^2(\pi c + \pi T\cos(\theta))\cosh^2(\pi T\sin(\theta)) + \cos^2(\pi c + \pi T\cos(\theta))\sinh^2(\pi T\sin(\theta))} \\
&= \sqrt{\sin^2(\pi c + \pi T\cos(\theta))(1+\sinh^2(\pi T\sin(\theta))) + (1-\sin^2(\pi c + \pi T\cos(\theta)))\sinh^2(\pi T\sin(\theta))} \\
&= \sqrt{\sin^2(\pi c + \pi T\cos(\theta)) + \sinh^2(\pi T\sin(\theta))}.
\end{aligned} \tag{7}$$

In conclusion:

$$\left|\sin(\pi(c+Te^{i\theta}))\right| = \sqrt{\sin^2(\pi c+\pi T\cos(\theta))+\sinh^2(\pi T\sin(\theta))} \geq \sqrt{\sinh^2(\pi T\sin(\theta))} = \sinh(\pi T|\sin(\theta)|). \tag{8}$$

Using this result in the estimates 3 we have:

$$\begin{aligned}
&\left|\int_{S_T}\frac{\pi}{\sin(\pi s)}\phi(-s)t^{-s}ds\right| < TC\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\frac{\pi}{\left|\sin(\pi(c+Te^{i\theta}))\right|}e^{AT|\sin(\theta)|}d\theta \\
&\leq TC\pi\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\frac{e^{AT|\sin(\theta)|}}{\sinh(\pi T|\sin(\theta)|)}d\theta = TC\pi\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\frac{2e^{AT|\sin(\theta)|}}{e^{\pi T|\sin(\theta)|}-e^{-\pi T|\sin(\theta)|}}d\theta \\
&= 2\pi CT\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\frac{e^{AT|\sin(\theta)|}}{e^{\pi T|\sin(\theta)|}-e^{-\pi T|\sin(\theta)|}}d\theta = 2\pi CT\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\frac{e^{-\pi T|\sin(\theta)|}}{e^{-\pi T|\sin(\theta)|}}\cdot\frac{e^{AT|\sin(\theta)|}}{e^{\pi T|\sin(\theta)|}-e^{-\pi T|\sin(\theta)|}}d\theta \\
&= 2\pi CT\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\frac{e^{(A-\pi)T|\sin(\theta)|}}{1-e^{-2\pi T|\sin(\theta)|}}d\theta = 2\pi CT\int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\frac{e^{-T(\pi-A)|\sin(\theta)|}}{1-e^{-2\pi T|\sin(\theta)|}}d\theta.
\end{aligned} \tag{9}$$

Consider now the succession of contours C_k composed by the straight line from $c-iT_k$ to $c+iT_k$ and the semicircle S_{T_k} with growing radii T_k satisfying $-\frac{2}{3}k \leq c-T_k \leq -\frac{1}{2}k$. This hypothesis on T_k ensures that, while traveling on these contours, the function $\frac{\pi}{\sin(\pi s)}$ avoids its singularities in $s=-k$ (see Figure 2).
Denote C' the limit contour as $k \to \infty$.

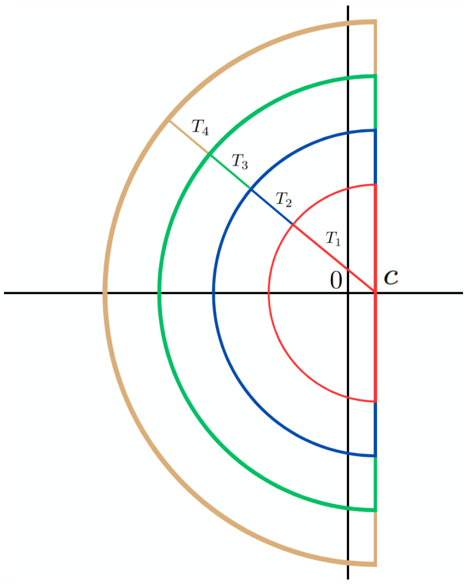

Figure 2: The Contour succession C_k

In this case, using the estimates 9:

$$\left|\int_{S_{T_k}} \frac{\pi}{\sin(\pi s)}\phi(-s)t^{-s}ds\right| < 2\pi CT \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \frac{e^{-T_k(\pi-A)|\sin(\theta)|}}{1-e^{-2\pi T_k|\sin(\theta)|}}d\theta \xrightarrow{k\to\infty} 0$$

due to the fact that, by hypothesis, $A < \pi$. Notice that $k \to \infty \Rightarrow T_k \to \infty$.

Therefore:

$$\frac{1}{2\pi i}\int_{C'} \frac{\pi}{\sin(\pi s)}\phi(-s)t^{-s}ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi}{\sin(\pi s)}\phi(-s)t^{-s}ds.$$

We can calculate the first integral using the Residue Theorem:
The function $\phi(s)$ is supposed analytic, so the only singularities contained in the contour C' are those of the function $\frac{\pi}{\sin(\pi s)}$, that is to say, the only singularities of the function in the interior of the domain are $s = 0, -1, -2, -3, \cdots$.
Those are simple poles, so the residues of the integrand can be computed easily:

$$Res\left(\frac{\pi}{\sin(\pi s)}\phi(-s)t^{-s}; -n\right) = \lim_{s\to -n} \frac{(s+n)\pi}{\sin(\pi s)}\phi(-s)t^{-s} = (-1)^n\phi(n)t^n.$$

Therefore:

$$\frac{1}{2\pi i}\int_{C'} \frac{\pi}{\sin(\pi s)}\phi(-s)t^{-s}ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi}{\sin(\pi s)}\phi(-s)t^{-s}ds = \sum_{n=0}^{\infty}\phi(n)(-t)^n = \Phi(t)$$

for any $0 < c < 1$.

We can now deduce 2 by using the well-know Mellin Inversion Formula:

Theorem 3 (Mellin Inversion formula)**.** *Assume that $F(s)$ is analytic in the strip $a < Re(s) < b$ and define f by:*

$$f(t) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)t^{-s}ds.$$

If this integral converges absolutely and uniformly for $c \in (a,b)$ then

$$F(s) = \int_0^{\infty} t^{s-1}f(t)dt. \tag{10}$$

In our case, we just proved that:

$$\Phi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi}{\sin(\pi s)}\phi(-s)t^{-s}ds$$

$$\Downarrow$$

$$\frac{\pi}{\sin(\pi s)}\phi(-s) = \int_0^{\infty} t^{s-1}\Phi(t)dt = \int_0^{\infty} t^{s-1}\sum_{n=0}^{\infty}\phi(n)(-t)^n dt.$$

□

Let's see how this Theorem 2 implies Theorem 1:

Proof of Ramanujan's Master Theorem:

Define the function:

$$\phi'(s) := \phi(s)\Gamma(1+s)$$

$$\Downarrow$$

$$\phi(s) = \frac{\phi'(s)}{\Gamma(1+s)}$$

and rewrite equation 2 using this new function:

$$\int_0^\infty t^{s-1} \sum_{n=0}^\infty \phi(n)(-t)^n dt = \frac{\pi}{\sin(s\pi)}\phi(-s)$$

$$\Downarrow$$

$$\int_0^\infty t^{s-1} \sum_{n=0}^\infty \frac{\phi'(n)}{\Gamma(1+n)}(-t)^n dt = \frac{\pi}{\sin(s\pi)} \frac{\phi'(s)}{\Gamma(1+s)}$$

$$\Downarrow$$

$$\int_0^\infty t^{s-1} \sum_{n=0}^\infty \frac{\phi'(n)}{n!}(-t)^n dt = \frac{\pi}{\sin(s\pi)} \frac{\phi'(s)}{\Gamma(1+s)}$$

due to the fact that $\Gamma(n+1) = n!$ for every $n \in \mathbb{N}$.

Using another known property of the Gamma Function:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

$$\Downarrow$$

$$\frac{\pi}{\sin(\pi s)} = \Gamma(-s)\Gamma(1+s)$$

(you can find on our site a demonstration of this Relation to the Sine Function.)

We finally have:

$$\int_0^\infty t^{s-1} \sum_{n=0}^\infty \frac{\phi'(n)}{n!}(-t)^n dt = \frac{\pi}{\sin(s\pi)} \frac{\phi'(s)}{\Gamma(1+s)}$$

$$\Downarrow$$

$$\int_0^\infty t^{s-1} \sum_{n=0}^\infty \frac{\phi'(n)}{n!}(-t)^n dt = \Gamma(-s)\phi'(s)$$

which proves Theorem 1. □

Thank you!

We hope this lesson has been beneficial in studying this interesting topic.
For more lessons or demonstrations, visit our website.

References

[1] Tewodros Amdeberhan et al. "Ramanujan's master theorem". In: *The Ramanujan Journal* 29 (2012), pp. 103–120.

[2] Godfrey Harold Hardy. *Ramanujan: twelve lectures on subjects suggested by his life and work*. Vol. 136. American Mathematical Soc., 1999.