The original purpose of defining the Riemann zeta, was to investigate the distribution of prime numbers, particularly the function $\pi(x)$, which counts the number of primes less than a given quantity.
Riemann's efforts in his 1859 article, "On the Number of Prime Numbers Less Than a Given Quantity," made significant waves in the world of mathematics. Still, he could not prove the conjecture initially stated by Legendre in 1797.

However, building on Riemann's theory of the Zeta function, Jacques Hadamard and Charles Jean de la Vallée-Poussin independently proved the conjecture in 1896.

Remark 1. *The following proof takes inspiration from the one appearing in [2].*

Theorem 1 (The Prime Number Theorem)**.** *Let $\pi(x)$ denote the number of primes $\leq x$. then:*

$$\pi(x) \sim \frac{x}{\log x}, \qquad \text{as } x \to \infty. \tag{1}$$

We will need the following lemma:

Lemma 1.

$$\log \zeta(s) = s \int_2^\infty \frac{\pi(x)}{x(x^s - 1)} \mathrm{d}x \tag{2}$$

for $\mathrm{Re}(s) > 1$.

Proof. Remember Euler's Product Formula for the Riemann Zeta function:

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} = \prod_p \left(1 - p^{-s}\right)^{-1} \tag{3}$$

where the product runs over all prime numbers p. This implies:

$$\log \zeta(s) = -\sum_p \log\left(1 - \frac{1}{p^s}\right). \tag{4}$$

While, the definition of $\pi(x)$ implies that:

$$\pi(n) - \pi(n-1) = \begin{cases} 1 & \text{if } n \text{ is a prime number} \\ 0 & \text{if } n \text{ is not a prime number.} \end{cases} \tag{5}$$

Hence, equation 4 can be written as:

$$\log\zeta(s) = -\sum_p \log\left(1-\frac{1}{p^s}\right) = -\sum_{n=2}^{\infty}(\pi(n)-\pi(n-1))\log\left(1-\frac{1}{n^s}\right). \tag{6}$$

Here $\pi(1) = 0$ while for every other n there are only two factors $\pi(n)$, one multiplied by $\log\left(1-\frac{1}{n^s}\right)$ and one multiplied by $-\log\left(1-\frac{1}{(n+1)^s}\right)$.
Therefore, we can write the series as:

$$\sum_{n=2}^{\infty}(\pi(n)-\pi(n-1))\log\left(1-\frac{1}{n^s}\right) = \sum_{n=2}^{\infty}\pi(n)\left(\log\left(1-\frac{1}{n^s}\right)-\log\left(1-\frac{1}{(n+1)^s}\right)\right). \tag{7}$$

It follows that we can develop further equation 6:

$$\begin{aligned}\log\zeta(s) &= -\sum_{n=2}^{\infty}(\pi(n)-\pi(n-1))\log\left(1-\frac{1}{n^s}\right)\\ &= -\sum_{n=2}^{\infty}\pi(n)\left(\log\left(1-\frac{1}{n^s}\right)-\log\left(1-\frac{1}{(n+1)^s}\right)\right)\\ &= \sum_{n=2}^{\infty}\pi(n)\int_n^{n+1}\frac{s}{x(x^s-1)}\mathrm{d}x = s\int_2^{\infty}\frac{\pi(x)}{x(x^s-1)}\mathrm{d}x\end{aligned} \tag{8}$$

where we are allowed to rearrange the series since:
$\pi(n) \leq n$ and $\log(1-n^{-s}) = \mathcal{O}(n^{-\mathrm{Re}(s)})$ with $\mathrm{Re}(s) > 1$. □

To obtain the result of the Prime Number Theorem, we will invert equation 2 into an explicit formula for $\pi(x)$.

Proof of the Prime Number Theorem:

Define $\omega(s)$ as follows:

$$\omega(s) := \int_2^{\infty}\frac{\pi(x)}{x^{s+1}(x^s-1)}\mathrm{d}x. \tag{9}$$

Then:

$$\frac{\log\zeta(s)}{s} - \omega(s) = \int_2^{\infty}\frac{\pi(x)}{x(x^s-1)}\mathrm{d}x - \int_2^{\infty}\frac{\pi(x)}{x^{s+1}(x^s-1)}\mathrm{d}x = \int_2^{\infty}\frac{\pi(x)}{x^{s+1}}\mathrm{d}x. \tag{10}$$

Let's take some time to analyze the function $\omega(s)$:

Since $\pi(x) \leq x$ the integral defining $\omega(s)$ converges uniformly, and is therefore regular and bounded, for $\mathrm{Re}(s) \geq \frac{1}{2}+\delta$, due to the fact that the integral

$$\int_2^{\infty}\frac{\pi(x)}{x^{\frac{1}{2}+\delta}(x^{\frac{1}{2}+\delta}-1)}\mathrm{d}x$$

converges uniformly.

This is similarly proven for $\omega'(s)$, since, computing the derivative under the integral sing:

$$\omega'(s) = \int_2^\infty \pi(x) \log x \frac{1 - 2x^s}{x^{s+1}(x^s - 1)^2} \mathrm{d}x.$$

Proceed by differentiating equation 10 with respect to s:

$$\frac{\log \zeta(s)}{s} - \omega(s) = \int_2^\infty \frac{\pi(x)}{x(x^s - 1)} \mathrm{d}x - \int_2^\infty \frac{\pi(x)}{x^{s+1}(x^s - 1)} \mathrm{d}x = \int_2^\infty \frac{\pi(x)}{x^{s+1}} \mathrm{d}x$$

$$\Downarrow$$

$$\frac{\frac{\zeta'(s)}{\zeta(s)} s - \log \zeta(s)}{s^2} - \omega'(s) = -\int_2^\infty \frac{\pi(x) \log x}{x^{s+1}} \mathrm{d}x$$

$$\Downarrow$$

$$-\frac{\zeta'(s)}{s\zeta(s)} + \frac{\log \zeta(s)}{s^2} + \omega'(s) = \int_2^\infty \frac{\pi(x) \log x}{x^{s+1}} \mathrm{d}x. \tag{11}$$

Call $\phi(s)$ the right-hand side of the equation, i.e.

$$\phi(s) := \int_2^\infty \frac{\pi(x) \log x}{x^{s+1}} \mathrm{d}x. \tag{12}$$

Define also:

$$g(x) := \int_0^x \frac{\pi(u) \log u}{u} \mathrm{d}u, \qquad h(x) := \int_0^x \frac{g(u)}{u} \mathrm{d}u. \tag{13}$$

Remark 2. *Mind that, by definition of $\pi(x)$, $\pi(x) = 0$ for $x < 2$. Hence, the same is true for $g(x)$ and $h(x)$.*

Integrating by parts $\phi(s)$, using the fact that $g'(x) = \frac{\pi(x) \log x}{x}$, we obtain:

$$\begin{aligned} \phi(s) &= \int_2^\infty \frac{\pi(x) \log x}{x^{s+1}} \mathrm{d}x = \int_0^\infty \frac{\pi(x) \log x}{x^{s+1}} \mathrm{d}x = \int_0^\infty g'(x) x^{-s} \mathrm{d}x \\ &= s \int_0^\infty g(x) x^{-s-1} \mathrm{d}x. \end{aligned} \tag{14}$$

Now, $h'(x) = \frac{g(x)}{x}$ and we can integrate by parts again:

$$\begin{aligned} \phi(s) &= s \int_0^\infty g(x) x^{-s-1} \mathrm{d}x = s \int_0^\infty h'(x) x^{-s} \mathrm{d}x \\ &= s^2 \int_0^\infty h(x) x^{-s-1} \mathrm{d}x \end{aligned} \tag{15}$$

$$\Downarrow$$

$$\phi(-s) = (-s)^2 \int_0^\infty h(x) x^{s-1} \mathrm{d}x$$

$$\Downarrow$$

$$\frac{\phi(-s)}{s^2} = \int_0^\infty h(x)x^{s-1}\mathrm{d}x. \tag{16}$$

Now $h(x)$ is continuous and of bounded variation in any finite interval and, since $\pi(x) \le x$, it follows that, for $x > 1$, $g(x) \le x\log x$ and $h(x) \le x\log x$.

In conclusion $h(x)x^{k-2}$ is absolutely integrable over $(0,\infty)$ if $k < 0$.

We can therefore use the inverse Mellin Transform to compute $\frac{h(x)}{x}$:

The **Mellin Transform** of a complex valued function $f(x)$ is the function:

$$\mathcal{M}\{f\}(s) = \int_0^\infty x^{s-1}f(x)\mathrm{d}x.$$

Therefore:

$$\frac{\phi(-s)}{s^2} = \mathcal{M}\left\{h(x)\right\}(s). \tag{17}$$

While the **inverse Mellin Transform** is:

$$\mathcal{M}^{-1}\{\varphi\}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\varphi(s)\mathrm{d}s$$

were c is a real value such that: $\phi(s) \to 0$ as $\mathrm{Im}(s) \to \pm\infty$ on the line $\mathrm{Re}(s) = c$, and the integral on the same line converges absolutely.

Therefore, for $c > 1$:

$$h(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\phi(-s)}{s^2}x^{-s}\mathrm{d}s = \frac{1}{2\pi i}\int_{c+i\infty}^{c-i\infty}\frac{\phi(s)}{s^2}x^{s}\left(-\mathrm{d}s\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\phi(s)}{s^2}x^{s}\mathrm{d}s \tag{18}$$

The integral on the right is absolutely convergent, since $\phi(s)$ is bounded for $\mathrm{Re}(s) \ge 1$, except in the neighborhood of $s = 1$

Remark 3. *These bounds are proven in detail in [2], page 51.*

In the neighborhood of $s = 1$, we have:

$$\phi(s) = \frac{1}{s-1} + \log\frac{1}{s-1} + \cdots$$

$$\Downarrow$$

$$\phi(s) = \frac{1}{s-1} + \psi(s)$$

where $\psi(s)$ is bounded for $\mathrm{Re}(s) \ge 1$, $|s-1| \ge 1$ and $\psi(s)$ has a logarithmic infinity as $s \to 1$.

Now

$$h(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\phi(s)}{s^2}x^{s}\mathrm{d}s = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s}}{(s-1)s^2}\mathrm{d}s + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\psi(s)}{s^2}x^{s}\mathrm{d}s.$$

The first term can be computed using the Residue Theorem, we can consider the contour C defined as the half-circle with side on the line $\mathrm{Re}(s) = c$ and extending to its left (see Figure 1).

It's easy to see that as the radius T goes to infinity the integral on the arc goes to zero and we are left with the term $\int_{c-i\infty}^{c+i\infty} \frac{x^s}{(s-1)s^2} \mathrm{d}s$. This is therefore equal to the sum of the residues on the left of the line $\mathrm{Re}(s) = c$.

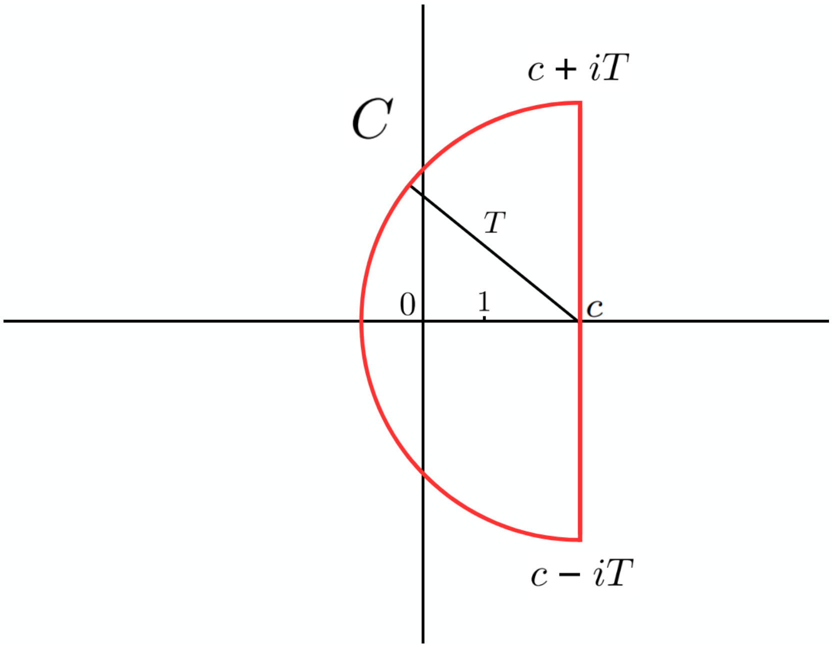

Figure 1: The contour C

In conclusion, the first term can be computed as:

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{(s-1)s^2} \mathrm{d}s = x - \log x - 1.$$

In the other term we can apply Cauchy's Theorem to the rectangle $(1 \pm iT, c \pm iT)$, with an indentation of radius ϵ around $s = 1$ and make $T \to \infty$, $\epsilon \to 0$ to basically compute the integral for $c = 1$ (see figure 2).

Hence:

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\psi(s)}{s^2} x^s \mathrm{d}s = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{\psi(s)}{s^2} x^s \mathrm{d}s = \frac{1}{2\pi i} \int_{\infty}^{\infty} \frac{\psi(1+it)}{(1+it)^2} x^{1+it} (i\mathrm{d}t)$$

$$\Downarrow$$

$$h(x) = x - \log x - 1 + \frac{x}{2\pi} \int_{-\infty}^{\infty} \frac{\psi(1+it)}{(1+it)^2} x^{it} \mathrm{d}t.$$

The last integral tends to zero as $x \to \infty$, by the extension to Fourier integrals of the Riemann-Lebesgue Theorem.

Remark 4. *This is Theorem 1 of [1].*

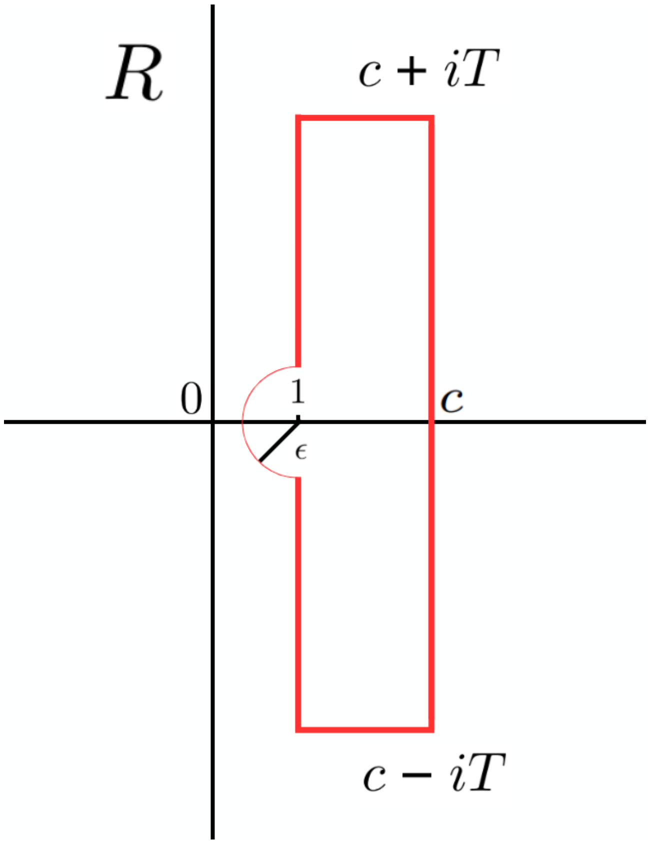

Figure 2: The contour R

Therefore:

$$h(x) \sim x.$$

To conclude, we need the following Lemma:

Lemma 2. *Let $f(x)$ be positive and non-decreasing, such that:*

$$\int_1^x \frac{f(u)}{u} \mathrm{d}u \sim x$$

as $x \to \infty$.

Then

$$f(x) \sim x.$$

Proof. Consider a given positive number δ, then by hypothesis we have:

$$(1-\delta)x < \int_1^x \frac{f(u)}{u} \mathrm{d}u < (1+\delta)x, \qquad (x > x_0(\delta)).$$

Hence, for any positive ϵ:

$$\int_x^{x(1+\epsilon)} \frac{f(u)}{u} \mathrm{d}u = \int_1^{x(1+\epsilon)} \frac{f(u)}{u} \mathrm{d}u - \int_1^x \frac{f(u)}{u} \mathrm{d}u < (1+\delta)(1+\epsilon)x - (1-\delta)x$$

$$\Downarrow$$

$$\int_x^{x(1+\epsilon)} \frac{f(u)}{u} \mathrm{d}u < (2\delta + \epsilon + \delta\epsilon)x.$$

But $f(x)$ is non-decreasing, so:

$$\int_x^{x(1+\epsilon)} \frac{f(u)}{u} \mathrm{d}u \geq f(x) \int_x^{x(1+\epsilon)} \frac{\mathrm{d}u}{u} > f(x) \int_x^{x(1+\epsilon)} \frac{\mathrm{d}u}{x(1+\epsilon)} = \frac{\epsilon}{1+\epsilon} f(x)$$

$$\Downarrow$$

$$\frac{\epsilon}{1+\epsilon} f(x) < \int_x^{x(1+\epsilon)} \frac{f(u)}{u} \mathrm{d}u < (2\delta + \epsilon + \delta\epsilon)x$$

$$\Downarrow$$

$$f(x) < x(1+\epsilon)\left(1 + \delta + \frac{2\delta}{\epsilon}\right).$$

Taking for example $\epsilon = \sqrt{\delta}$, it follows that:

$$\limsup \frac{f(x)}{x} \leq 1.$$

With the same reasoning, considering the integral:

$$\int_{x(1-\epsilon)}^x \frac{f(u)}{u} \mathrm{d}u$$

we obtain

$$\liminf \frac{f(x)}{x} \geq 1$$

and the lemma follows. □

We have proven that $h(x) \sim x$ hence, this lemma implies:

$$h(x) = \int_1^x \frac{g(u)}{u} \mathrm{d}u \sim x$$

$$\Downarrow$$

$$g(x) \sim x$$

$$\Downarrow$$

$$g(x) = \int_1^x \frac{\pi(u) \log u}{u} \mathrm{d}u \sim x$$

$$\Downarrow$$

$$\pi(x) \log x \sim x$$

which proves 1. □

POSITIVE
INCREMENT

Thank you!

We hope this lesson has been beneficial in studying this interesting topic.
For more lessons or demonstrations, visit our website.

References

[1] Edward Charles Titchmarsh. *Introduction to the theory of Fourier integral.* The Clarendon Press, 1937.

[2] Edward Charles Titchmarsh and David Rodney Heath-Brown. *The theory of the Riemann zeta-function.* Oxford university press, 1986.