POSITIVE
INCREMENT

Attempting to prove the Riemann Hypothesis directly seems to be hopeless with our current methods. It is therefore essential to know that a variety of different statements are equivalent to the Hypothesis and might represent a path to prove it indirectly.

In this lesson, we explain in detail an equivalence Theorem, first proved by Riesz in 1916.

Remark 1. *The following lesson is inspired by chapter* 14 *of [2], the original proof of the final result comes from [1].*

Definition 1. *We start by defining the function* $F(x)$ *as:*

$$F(x) := \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{(k-1)!\zeta(2k)} \tag{1}$$

We will firstly use the Residue Theorem to prove that:

$$F(x) = \frac{i}{2} \int_{a-i\infty}^{a+i\infty} \frac{x^s}{\Gamma(s)\zeta(2s)\sin(\pi s)} \mathrm{d}s = \frac{i}{2\pi} \int_{a-i\infty}^{a+i\infty} \frac{\Gamma(1-s)}{\zeta(2s)} x^s \mathrm{d}s. \tag{2}$$

Consider the integral:

$$\int_{S_T} \frac{x^s}{\Gamma(s)\zeta(2s)\sin(\pi s)} \mathrm{d}s \tag{3}$$

where S_T is the semicircle consisting of the vertical line segment from $a - iT$ to $a + iT$ with $\frac{1}{2} < a < 1$, and a semicircular arc of radius T connecting $a - iT$ to $a + iT$ from the right side of the complex plane. S_T is traveled clockwise (see figure 1).

Divide the integral on the line segment plus the integral on the semicircular arc, that we will denote C_T:

$$\int_{S_T} \frac{x^s}{\Gamma(s)\zeta(2s)\sin(\pi s)} \mathrm{d}s = \int_{a-iT}^{a+iT} \frac{x^s}{\Gamma(s)\zeta(2s)\sin(\pi s)} \mathrm{d}s + \int_{C_T} \frac{x^s}{\Gamma(s)\zeta(2s)\sin(\pi s)} \mathrm{d}s.$$

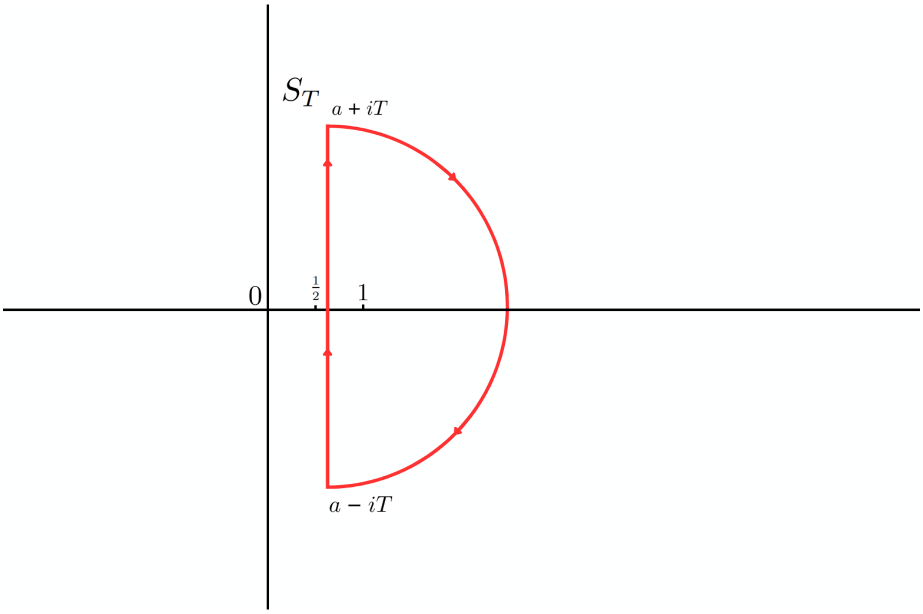

Figure 1: The Contour S_T

We will prove that the integral on the arc goes to zero as $T \to \infty$.

To do this notice that:

$$\left|\frac{x^s}{\Gamma(s)\zeta(2s)\sin(\pi s)}\right| = \left|\frac{x^{\sigma+iT}}{\Gamma(s)\zeta(2s)\sin(\pi s)}\right| = x^\sigma \left|\frac{e^{iT\log x}}{\Gamma(s)\zeta(2s)\sin(\pi s)}\right|.$$

For $s \in C_T$, $\zeta(2s) \neq 0$ as we are outside the critical strip. $e^{iT\log x}$ can be simply estimated as:

$$e^{iT\log x} = \cos(T\log x) + i\sin(T\log x).$$

While Stirling's approximation formula states that, for large enough s:

$$\Gamma(s) = \sqrt{\frac{2\pi}{s}}\left(\frac{s}{e}\right)^s\left(1 + \mathcal{O}\left(\frac{1}{s}\right)\right).$$

With these estimates, it is easy to conclude that as the absolute value of s grows in the arc C_T the argument of the integral is going to zero, that is to say:

$$\lim_{T\to\infty}\int_{S_T}\frac{x^s}{\Gamma(s)\zeta(2s)\sin(\pi s)}\mathrm{d}s = \int_{a-i\infty}^{a+i\infty}\frac{x^s}{\Gamma(s)\zeta(2s)\sin(\pi s)}\mathrm{d}s.$$

We will now compute the integral on the left using the Residue Theorem.

First remember the known relation between the Gamma and Sine Function:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

$$\Downarrow$$

$$\Gamma(1-s) = \frac{\pi}{\Gamma(s)\sin(\pi s)}.$$

Therefore:

$$\int_{S_T} \frac{x^s}{\Gamma(s)\zeta(2s)\sin(\pi s)} \mathrm{d}s = \frac{1}{\pi}\int_{S_T} \frac{\Gamma(1-s)}{\zeta(2s)} x^s \mathrm{d}s. \tag{4}$$

By the Residue Theorem we know that:

$$\frac{1}{2\pi i}\int_{S_T} \frac{\Gamma(1-s)}{\zeta(2s)} x^s \mathrm{d}s = -\sum \mathrm{Res}\left(\frac{\Gamma(1-s)}{\zeta(2s)} x^s\right)$$

where the sum runs over all the residues contained in the contour and the minus sign is due to the fact that the contour S_T is traveled clockwise.

Therefore

$$\int_{S_T} \frac{x^s}{\Gamma(s)\zeta(2s)\sin(\pi s)} \mathrm{d}s = \frac{1}{\pi}\int_{S_T} \frac{\Gamma(1-s)}{\zeta(2s)} x^s \mathrm{d}s = -2i\sum \mathrm{Res}\left(\frac{\Gamma(1-s)}{\zeta(2s)} x^s\right).$$

The only poles in the contour are those coming from $\Gamma(1-s)$ for $s = 1, 2, 3, \cdots$ where it has simple poles. Let's calculate the Residue at a generic pole $s = k$ for $k \geq 1$:

$$\mathrm{Res}_{s=k}\left(\frac{\Gamma(1-s)}{\zeta(2s)} x^s\right) = \frac{\mathrm{Res}_{s=k}\left(\Gamma(1-s)\right)}{\zeta(2k)} x^k = \frac{\mathrm{Res}_{s=k}\left(\Gamma(-(s-1))\right)}{\zeta(2k)} x^k = \frac{(-1)^{k-1} x^k}{(k-1)!\zeta(2k)}$$

where we used the known fact that the residue of the Gamma Function in $s = -n$ is $\frac{(-1)^n}{n!}$.

Hence:

$$\int_{S_T} \frac{x^s}{\Gamma(s)\zeta(2s)\sin(\pi s)} \mathrm{d}s = -2i\sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{(k-1)!\zeta(2k)}$$

$$\Downarrow$$

$$\frac{i}{2}\int_{a-i\infty}^{a+i\infty} \frac{x^s}{\Gamma(s)\zeta(2s)\sin(\pi s)} \mathrm{d}s = \frac{i}{2}\int_{S_T} \frac{x^s}{\Gamma(s)\zeta(2s)\sin(\pi s)} \mathrm{d}s = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{(k-1)!\zeta(2k)} = F(x)$$

this combined with equation 4 proves 2.

We can now prove that:

Theorem 1. *The Riemann Hypothesis is true if and only if*

$$F(x) = \mathcal{O}\left(x^{\frac{1}{4}+\epsilon}\right).$$

Proof. Choose a slightly bigger than $\frac{1}{2}$, say $a = \frac{1}{2} + \epsilon$ with $\epsilon << 1$, then, changing variable to $s = a + it$ we have:

$$\frac{i}{2\pi}\int_{a-i\infty}^{a+i\infty}\frac{\Gamma(1-s)}{\zeta(2s)}x^{s}\mathrm{d}s = \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma(1-a+it)}{\zeta(2(a+it))}x^{a+it}i\mathrm{d}t = -\frac{x^{a}}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma(1-a+it)}{\zeta(2(a+it))}e^{it\log x}\mathrm{d}t.$$

Similarly to what we did before, the integral is bounded in the integration path, therefore:

$$F(x) = \mathcal{O}(x^{a}) = \mathcal{O}\left(x^{\frac{1}{2}+\epsilon}\right).$$

If we suppose true the Riemann Hypothesis, then we can move the line of integration to $a = \frac{1}{4} + \epsilon$ without enclosing any pole, we would therefore have the same formulas, this time yielding:

$$F(x) = \mathcal{O}(x^{a}) = \mathcal{O}\left(x^{\frac{1}{4}+\epsilon}\right). \tag{5}$$

Conversely, Mellin's inversion Formula implies that:

$$\frac{\Gamma(1-s)}{\zeta(2s)} = -\int_{0}^{\infty}F(x)x^{-1-s}\mathrm{d}s \tag{6}$$

If condition 5 where to be true, then the integral converges uniformly for $\mathrm{Re}(s) \geq \sigma_0 > \frac{1}{4}$, therefore the analytic function on the left of the equation is regular for $\mathrm{Re}(s) > \frac{1}{4}$ and the truth of the Riemann Hypothesis follows. ☐

References

[1] Marcel Riesz, Garding Lars, and Lars Hörmander. "Sur L'hypothese de Riemann". In: *Collected Papers*. Springer, 1988, pp. 165–170.

[2] Edward Charles Titchmarsh and David Rodney Heath-Brown. *The theory of the Riemann zeta-function*. Oxford university press, 1986.