POSITIVE INCREMENT

In 1859, Bernhard Riemann published the article titled "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse," [3] which translates to "On the Number of Prime Numbers less than a Given Quantity" [2]. This work laid the foundation for centuries of future research.

In this initial analysis, we will focus on the first few pages of the article, which lead to the statement of the Hypothesis.

1 Definition of the Riemann Zeta and its first functional equation

The article starts by defining the now famous **Riemann Zeta function** $\zeta(s)$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and recalling a property already known to Euler:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}} \tag{1}$$

where the product is over every prime number p. Riemann provides no proof for this already well-established result, but you can find the details on our site.

Both expressions only converge when $Re(s) > 1$; the first pages of the paper are dedicated to finding different expressions of the function that remain valid for any $s \neq 1$.

First Riemann notices that:

$$\int_0^{\infty} x^{s-1} e^{-nx} dx = \int_0^{\infty} \left(\frac{y}{n}\right)^{s-1} e^{-y} \frac{dy}{n} = \frac{1}{n^s} \int_0^{\infty} y^{s-1} e^{-y} dy = \frac{\Gamma(s)}{n^s}$$

here he used the definition of the Gamma function: $\Gamma(s) := \int_0^{\infty} y^{s-1} e^{-y} dy$.

Remark 1.1. *In his paper Riemann uses the notation $\Pi(s)$ to indicate the Gamma function, we will always use $\Gamma(s)$ in this analysis; if one wishes to follow this side to side with the original, mind that $\Gamma(s) = \Pi(s-1)$.*

Therefore, summing both sides from $n = 1$ to ∞ yields:

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1}. \tag{2}$$

Equation 2 is one of the first examples that illustrates the relationship between the Gamma function and the Riemann Zeta function, Visit our website for more details.

The author considers now the integral:

$$\int_C \frac{(-x)^{s-1}}{e^x - 1} dx$$

where C is a contour that starts from $+\infty$, circles around the value 0 containing no other point of discontinuity of the integrand in its interior, let's call the radius of this circle δ, and then goes back to $+\infty$ (see Figure 1).

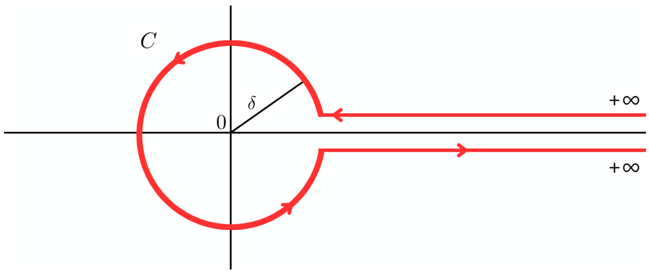

Figure 1: The contour C

Now to simplify this integral, we write $(-x)^{s-1} = e^{(s-1)\log(-x)}$, making first sure that $\log(-x)$ is well defined for $x > 0$:

Using the fact that $-1 = e^{i\pi} = e^{-i\pi}$ we can write:

$$\log(-x) = \log(e^{i\pi}x) = i\pi + \log(x)$$

and

$$\log(-x) = \log(e^{-i\pi}x) = -i\pi + \log(x).$$

However we must make choices that preserve the continuity of $\log(-x)$ as x moves along the contour in the complex plane, Riemann also specifies that we want $\log(-x)$ to be real when x is negative.

Therefore we must choose $\log(-x) = -i\pi + \log(x)$ for the piece of the contour from $+\infty$ to 0 and to preserve continuity, $\log(-x) = i\pi + \log(x)$ on the way back from 0 to $+\infty$ (see Figure 2).

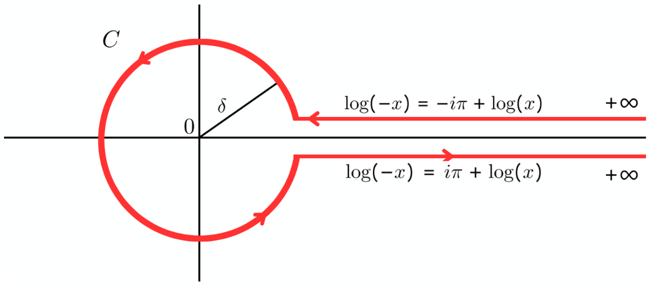

Figure 2: Appropriate choices of the logarithmic function

Remark 1.2. *This type of contour integrals are widespread in Analytic Number Theory, often called* ***Hankel's loop integrals.***

Thus the integral becomes:

$$
\begin{aligned}
&\int_C \frac{(-x)^{s-1}}{e^x-1}dx = \int_{+\infty}^{\delta} \frac{e^{(s-1)\log(-x)}}{e^x-1}dx + \int_{\delta}^{+\infty} \frac{e^{(s-1)\log(-x)}}{e^x-1}dx + \int_{|x|=\delta} \frac{(-x)^{s-1}}{e^x-1}dx \\
&= \int_{+\infty}^{\delta} \frac{e^{(s-1)(-i\pi+\log(x))}}{e^x-1}dx + \int_{\delta}^{+\infty} \frac{e^{(s-1)(i\pi+\log(x))}}{e^x-1}dx + \int_{|x|=\delta} \frac{(-x)^{s-1}}{e^x-1}dx \\
&= e^{(s-1)(-i\pi)} \int_{+\infty}^{\delta} \frac{x^{s-1}}{e^x-1}dx + e^{(s-1)(i\pi)} \int_{\delta}^{+\infty} \frac{x^{s-1}}{e^x-1}dx + \int_{|x|=\delta} \frac{(-x)^{s-1}}{e^x-1}dx \\
&= e^{-i\pi s} \cdot e^{i\pi} \int_{+\infty}^{\delta} \frac{x^{s-1}}{e^x-1}dx + e^{i\pi s} \cdot e^{-i\pi} \int_{\delta}^{+\infty} \frac{x^{s-1}}{e^x-1}dx + \int_{|x|=\delta} \frac{(-x)^{s-1}}{e^x-1}dx \\
&= -e^{-i\pi s} \int_{+\infty}^{\delta} \frac{x^{s-1}}{e^x-1}dx - e^{i\pi s} \int_{\delta}^{+\infty} \frac{x^{s-1}}{e^x-1}dx + \int_{|x|=\delta} \frac{(-x)^{s-1}}{e^x-1}dx \\
&= e^{-i\pi s} \int_{\delta}^{+\infty} \frac{x^{s-1}}{e^x-1}dx - e^{i\pi s} \int_{\delta}^{+\infty} \frac{x^{s-1}}{e^x-1}dx + \int_{|x|=\delta} \frac{(-x)^{s-1}}{e^x-1}dx \\
&= (e^{-i\pi s} - e^{i\pi s}) \int_{\delta}^{+\infty} \frac{x^{s-1}}{e^x-1}dx + \int_{|x|=\delta} \frac{(-x)^{s-1}}{e^x-1}dx.
\end{aligned} \tag{3}
$$

Notice now that, on the circle $|x| = \delta$ we have, denoting $s = \sigma + it$:

$$
\begin{aligned}
&|(-x)^{s-1}| = |x^{s-1}| = |e^{\log(x)(s-1)}| = |e^{(\log(|x|)+i\arg(x))(\sigma-1+it)}| \\
&= |e^{\log(|x|)(\sigma-1)-t\arg(x)} \cdot e^{i(\log(|x|)t+\arg(x)\sigma-1)}| \\
&= e^{(\sigma-1)\log(|x|)-t\arg(x)} \le |x|^{\sigma-1}e^{2\pi|t|}
\end{aligned} \tag{4}
$$

and

$$|e^x - 1| > A|x|$$

for an adequate constant A.

Therefore the integral around the circle can be estimated by:

$$\left| \int_{|x|=\delta} \frac{(-x)^{s-1}}{e^x - 1} dx \right| \leq \int_{|x|=\delta} \left| \frac{(-x)^{s-1}}{e^x - 1} \right| dx \leq \int_{|x|=\delta} \frac{|x|^{\sigma-1} e^{2\pi|t|}}{A|x|} dx = \frac{2\pi e^{2\pi|t|}}{A} \delta^{\sigma-2}. \tag{5}$$

Hence the integral tends to zero with δ if $\sigma > 2$.

Thus, considering $\delta \to 0$ in equation 3, we have derived the first formula presented by Riemann:

$$\int_C \frac{(-x)^{s-1}}{e^x - 1} dx = (e^{-i\pi s} - e^{i\pi s}) \int_0^{+\infty} \frac{x^{s-1}}{e^x - 1} dx \tag{6}$$

remembering that $\sin(\pi s) = \frac{e^{i\pi s} - e^{-i\pi s}}{2i} \Rightarrow -2i \sin(\pi s) = (e^{-i\pi s} - e^{i\pi s})$ we find:

$$\int_C \frac{(-x)^{s-1}}{e^x - 1} dx = -2i \sin(\pi s) \int_0^{+\infty} \frac{x^{s-1}}{e^x - 1} dx$$

$$\Downarrow$$

$$i \int_C \frac{(-x)^{s-1}}{e^x - 1} dx = 2 \sin(\pi s) \int_0^{+\infty} \frac{x^{s-1}}{e^x - 1} dx$$

therefore, using equation 2:

$$2 \sin(\pi s) \Gamma(s) \zeta(s) = i \int_C \frac{(-x)^{s-1}}{e^x - 1} dx. \tag{7}$$

2 Consequences of this functional equation

With this formula, Riemann notices two important results:

First, the integral is a well defined function of s. This is due to the fact that e^x grows much faster than x^{s-1} as $x \to \infty$. Therefore, the integral converges uniformly in any finite region of the s-plane.
As a result, **the equation provides the value of the function $\zeta(s)$ for all complex values of s, demonstrating that the function is single-valued and finite for all finite values of s, except at $s = 1$.**

Second, Riemann claims that $\zeta(s) = 0$ **when s is a negative even integer;** these are known as the **Trivial Zeros**.
The validity of this statement is not as straightforward as it may seem; we will take some time to explain it:

Start by using the Relation to the Sine Function for the Gamma Function:

$$\Gamma(s)\Gamma(-s) = -\frac{\pi}{s \sin(\pi s)} \tag{8}$$

$$\Downarrow$$

$$\Gamma(s) \sin(\pi s) = -\frac{\pi}{s\Gamma(-s)}$$

so that equation 7 can be written as:

$$-\frac{2\pi}{s\Gamma(-s)}\zeta(s) = i\int_C \frac{(-x)^{s-1}}{e^x - 1}dx$$
$$\Downarrow$$
$$\zeta(s) = i\frac{-s\Gamma(-s)}{2\pi}\int_C \frac{(-x)^{s-1}}{e^x-1}dx = i\frac{\Gamma(1-s)}{2\pi}\int_C \frac{(-x)^{s-1}}{e^x-1}dx. \tag{9}$$

To work on this integral, remember that the function $\frac{x}{e^x-1}$ is analytic near $x = 0$ and can therefore be expanded as

$$\frac{x}{e^x-1} = \sum_{m=0}^{\infty}\frac{B_m x^m}{m!}$$

where the coefficients B_m are by definition the **Bernoulli numbers**. This expansion is valid in the disk $|x| < 2\pi$.

With this expansion, equation 9, with $\delta < 2\pi$ and $s = -n$, $n = 0, 1, 2, \cdots$ becomes:

$$\begin{aligned}
\zeta(-n) &= i\frac{\Gamma(n+1)}{2\pi}\int_C \frac{(-x)^{-n-1}}{e^x-1}dx = i\frac{\Gamma(n+1)}{2\pi}\left[(e^{i\pi n} - e^{-i\pi n})\int_\delta^{+\infty}\frac{x^{-n-1}}{e^x-1}dx + \int_{|x|=\delta}\frac{(-x)^{-n-1}}{e^x-1}dx\right] \\
&= i\frac{\Gamma(n+1)}{2\pi}\int_{|x|=\delta}(-x)^{-n-2}\frac{-x}{e^x-1}dx = -i\frac{\Gamma(n+1)}{2\pi}\int_{|x|=\delta}(-x)^{-n-2}\frac{x}{e^x-1}dx \\
&= \frac{\Gamma(n+1)}{2\pi i}\int_{|x|=\delta}(-x)^{-n-2}\left(\sum_{m=0}^{\infty}\frac{B_m x^m}{m!}\right)dx \\
&= \sum_{m=0}^{\infty}\Gamma(n+1)\frac{B_m}{m!}(-1)^n\frac{1}{2\pi i}\int_{|x|=\delta}x^{m-n-2}dx
\end{aligned} \tag{10}$$

we are allowed to switch the integral and the series thanks to Lebesgue's dominated convergence theorem.

The last integral can be calculated using the residue theorem, we have to separate 3 possible cases:

$$\begin{cases}
m > n+1 \Longrightarrow \text{The function is holomorphic} \Longrightarrow \int_{|x|=\delta}x^{m-n-2}dx = 0. \\
m = n+1 \Longrightarrow \text{The residue of the function in } x = 0 \text{ is } 1 \Longrightarrow \int_{|x|=\delta}x^{-1}dx = 2\pi i. \\
m < n+1 \Longrightarrow \text{The residue of the function in } x = 0 \text{ is } 0 \Longrightarrow \int_{|x|=\delta}x^{m-n-2}dx = 0.
\end{cases}$$

Therefore:

$$\zeta(-n) = \sum_{m=0}^{\infty}\Gamma(n+1)\frac{B_m}{m!}(-1)^n\frac{1}{2\pi i}\int_{|x|=\delta}x^{m-n-2}dx = n!\frac{B_{n+1}}{(n+1)!}(-1)^n$$
$$\Downarrow$$
$$\zeta(-n) = (-1)^n\frac{B_{n+1}}{n+1}. \tag{11}$$

It is a known property of the Bernoulli Numbers that B_k is zero when k is an odd number bigger than 1, therefore equation 11 proves Riemann's statement.

3 The second functional equation

The second functional equation presented in the paper is derived from the same integral, this time for $Re(s) = \sigma < 0$. Instead of being evaluated in a positive sense around the specified domain, the integral is evaluated in a negative sense around a new domain C_Δ. This domain consists of a larger circle, with a radius $\Delta > \delta$, which we will refer to as S_Δ. We then intersect this larger circle with a section of the original contour C (see Figure 3).

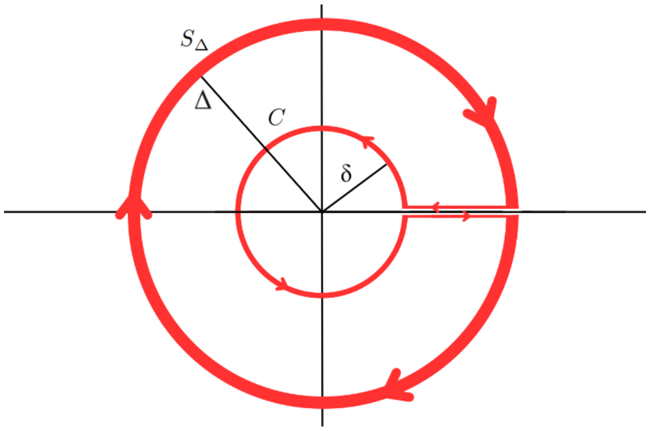

Figure 3: The contour C_Δ

We will prove that the integral on the external circle goes to 0 as the radius goes to infinity so that, denoting C' the limit contour for $\Delta \to +\infty$:

$$\int_{C'} \frac{(-x)^{s-1}}{e^x - 1} dx = \int_C \frac{(-x)^{s-1}}{e^x - 1} dx.$$

First notice that on the sliced circle S_Δ we have $-x = \Delta \cdot e^{i\psi}$ with $-\pi < \psi < \pi$ then, writing again $s = \sigma + it$:

$$|(-x)^{s-1}| = \Delta^{\sigma-1}$$

choose now Δ big enough so that $e^x - 1$ remains bounded away from 0 on that part of the contour, i.e. $|e^x - 1| \geq c$ for some $c > 0$, we find:

$$\left| \int_{S_\Delta} \frac{(-x)^{s-1}}{e^x - 1} dx \right| \leq \int_{S_\Delta} \left| \frac{(-x)^{s-1}}{e^x - 1} \right| dx \leq 2\pi \frac{\Delta^{\sigma-1}}{c}$$

when $\sigma < 0$ this tends to 0 for $\Delta \to +\infty$.

Now the integral $\int_{C'} \frac{(-x)^{s-1}}{e^x-1} dx$ can be calculated using the Residue theorem. In the interior of the domain the integrand has discontinuities only where x becomes equal to a whole multiple of $\pm 2\pi i$ and $Res\left(\frac{(-x)^{s-1}}{e^x-1}, 2n\pi i\right) = (-n2\pi i)^{s-1}$.

Therefore:

$$\begin{aligned}
\int_{C'} \frac{(-x)^{s-1}}{e^x - 1} dx &= -2\pi i \left[\sum_{n \in \mathbb{Z}^*} (-n2\pi i)^{s-1} \right] \\
&= \left[\sum_{n \geq 1} (-n2\pi i)^{s-1}(-2\pi i) + (n2\pi i)^{s-1}(-2\pi i) \right] \\
&= (2\pi)^s \sum_{n \geq 1} n^{s-1}((-i)^s + i^{s-2}).
\end{aligned} \tag{12}$$

Hence, equation 7 can be written as:

$$2\sin(\pi s)\Gamma(s)\zeta(s) = i\int_C \frac{(-x)^{s-1}}{e^x - 1} dx = i\int_{C'} \frac{(-x)^{s-1}}{e^x - 1} dx = i \cdot (2\pi)^s \sum_{n \geq 1} n^{s-1}((-i)^s + i^{s-2}). \tag{13}$$

We have proven the second functional equation in the paper:

$$2\sin(\pi s)\Gamma(s)\zeta(s) = (2\pi)^s \sum_{n \geq 1} n^{s-1}((-i)^{s-1} + i^{s-1}). \tag{14}$$

Notice that $n^{s-1} = \frac{1}{n^{1-s}} \Rightarrow \sum_{n \geq 1} n^{s-1} = \zeta(1-s)$ by definition of ζ.
Equation 14 is therefore a relation between $\zeta(s)$ and $\zeta(1-s)$.

Riemann asserts that by utilizing certain properties of the Gamma function, we can derive a more streamlined version of this relationship. We include some necessary details:

Firstly, we just noticed that equation 14 can be written as:

$$2\sin(\pi s)\Gamma(s)\zeta(s) = \zeta(1-s)(2\pi)^s((-i)^{s-1} + i^{s-1}).$$

Remembering that $\log(i) = i\frac{\pi}{2}$ we can write $i^{s-1} = e^{\log(i)(s-1)} = e^{i\pi\frac{(s-1)}{2}}$ and $(-i)^{s-1} = (i^{-1})^{s-1} = e^{-i\pi\frac{(s-1)}{2}}$, therefore:

$$2\sin(\pi s)\Gamma(s)\zeta(s) = \zeta(1-s)(2\pi)^s(e^{-i\pi\frac{(s-1)}{2}} + e^{i\pi\frac{(s-1)}{2}})$$

$$\Downarrow$$

$$2\sin(\pi s)\Gamma(s)\zeta(s) = \zeta(1-s)(2\pi)^s 2\cos\left(\pi\frac{(s-1)}{2}\right).$$

Using the known trigonometric identities:

$$\cos(-\theta) = \cos(\theta) \qquad\qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$$

we have:

$$\cos\left(\pi\frac{(s-1)}{2}\right) = \cos\left(\pi\frac{(1-s)}{2}\right) = \cos\left(\frac{\pi}{2} - s\frac{\pi}{2}\right) = \sin\left(s\frac{\pi}{2}\right) \tag{15}$$

therefore:

$$\sin(\pi s)\Gamma(s)\zeta(s) = \zeta(1-s)(2\pi)^s \sin\left(s\frac{\pi}{2}\right). \tag{16}$$

Proceed by using again the Relation to the Sine Function for the Gamma Function, which also implies:

$$\sin(\pi s) = \frac{\pi}{\Gamma(s)\Gamma(1-s)} \tag{17}$$

so equation 16 becomes:

$$\frac{\pi}{\Gamma(s)\Gamma(1-s)}\Gamma(s)\zeta(s) = \zeta(1-s)(2\pi)^s \sin\left(\frac{\pi s}{2}\right)$$
$$\Downarrow$$
$$\zeta(s) = \pi^{-1}\zeta(1-s)(2\pi)^s \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s). \tag{18}$$

Using again 17 we have:

$$\sin\left(\frac{\pi s}{2}\right) = \frac{\pi}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}. \tag{19}$$

Consider now Legendre's duplication formula for the Gamma function:

$$2\pi^{\frac{1}{2}}2^{-2s}\Gamma(2s) = \Gamma(s)\Gamma\left(s+\frac{1}{2}\right)$$

replacing s by $\frac{1-s}{2}$, this becomes:

$$2\pi^{\frac{1}{2}}2^{s-1}\Gamma(1-s) = \Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1-s+1}{2}\right)$$
$$\Downarrow$$
$$\Gamma(1-s) = 2^{-s}\pi^{-\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1-\frac{s}{2}\right). \tag{20}$$

Multiplying 20 and 19 we have:

$$\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) = 2^{-s}\pi^{-\frac{1}{2}}\cdot\pi\frac{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}$$
$$\Downarrow$$
$$\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) = 2^{-s}\pi^{\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.$$

Therefore, replacing $\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)$ in equation 18:

$$\zeta(s) = \zeta(1-s)(2\pi)^s 2^{-s}\pi^{-1}\pi^{\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$$
$$\Downarrow$$
$$\zeta(s)\Gamma\left(\frac{s}{2}\right) = \zeta(1-s)\Gamma\left(\frac{1-s}{2}\right)\pi^{s-\frac{1}{2}}$$
$$\Downarrow$$
$$\pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{s}{2}\right) = \zeta(1-s)\Gamma\left(\frac{1-s}{2}\right)\pi^{\frac{s-1}{2}}. \tag{21}$$

This proves that:

The expression $\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}$ is unchanged when s is replaced by $1-s$.

4 Definition of Riemann's ξ Function

In the succeeding paragraph, Riemann obtains a very elegant expression for equation 21.

First he uses the fact that:

$$\int_0^\infty x^{\frac{s}{2}-1} e^{-n^2\pi x} dx = \int_0^\infty \left(\frac{y}{n^2\pi}\right)^{\frac{s}{2}-1} e^{-y} \frac{dy}{n^2\pi} = \frac{1}{n^s \pi^{\frac{s}{2}}} \int_0^\infty \frac{y^{\frac{s}{2}-1}}{e^y} dy = \frac{\Gamma\left(\frac{s}{2}\right)}{n^s \pi^{\frac{s}{2}}}$$

where he used again the definition of the Gamma function.

He then defines:

$$\psi(x) := \sum_{n=1}^{\infty} e^{-n^2\pi x}$$

and sums both sides of the equality, supposing $\sigma > 1$, to obtain:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}} = \frac{\Gamma\left(\frac{s}{2}\right)\zeta(s)}{\pi^{\frac{s}{2}}} = \sum_{n=1}^{\infty} \int_0^\infty x^{\frac{s}{2}-1} e^{-n^2\pi x} dx = \int_0^\infty x^{\frac{s}{2}-1} \cdot \left(\sum_{n=1}^{\infty} e^{-n^2\pi x}\right) dx \tag{22}$$

where the inversion of the order of summation and integration is justified by absolute convergence as, for $\sigma > 1$:

$$\sum_{n=1}^{\infty} \int_0^\infty x^{\frac{\sigma}{2}-1} e^{-n^2\pi x} dx = \frac{\Gamma\left(\frac{\sigma}{2}\right)\zeta(\sigma)}{\pi^{\frac{1}{2}\sigma}}$$

converges.

Therefore:

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_0^\infty \psi(x) x^{\frac{s}{2}-1} dx. \tag{23}$$

The function $\psi(x)$ satisfies the equation:

$$2\psi(x) + 1 = x^{-\frac{1}{2}}\left(2\psi\left(\frac{1}{x}\right) + 1\right) \tag{24}$$

Riemann assumes this nontrivial fact, but a rigorous proof can be found on our site. Note that in the referred text, the function is denoted $\omega(x)$.

We can use 24 to elaborate 23:

$$
\begin{aligned}
\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} &= \int_0^{\infty} x^{\frac{s}{2}-1}\psi(x)dx = \int_0^1 x^{\frac{s}{2}-1}\psi(x)dx + \int_1^{\infty} x^{\frac{s}{2}-1}\psi(x)dx \\
&= \int_0^1 x^{\frac{s}{2}-1}\left[\frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2}\right]dx + \int_1^{\infty} x^{\frac{s}{2}-1}\psi(x)dx \\
&= \int_0^1 \frac{x^{\frac{s}{2}-1}}{\sqrt{x}}\psi\left(\frac{1}{x}\right)dx + \int_0^1 \frac{x^{\frac{s}{2}-1}}{2\sqrt{x}}dx - \frac{1}{2}\int_0^1 x^{\frac{s}{2}-1}dx + \int_1^{\infty} x^{\frac{s}{2}-1}\psi(x)dx \\
&= \int_0^1 x^{\frac{s}{2}-1}\frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right)dx + \frac{1}{2}\int_0^1 x^{\frac{s}{2}-\frac{3}{2}}dx - \frac{1}{s} + \int_1^{\infty} x^{\frac{s}{2}-1}\psi(x)dx \\
&= \int_0^1 x^{\frac{s}{2}-1}\frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right)dx + \frac{1}{s-1} - \frac{1}{s} + \int_1^{\infty} x^{\frac{s}{2}-1}\psi(x)dx.
\end{aligned}
\tag{25}
$$

Changing variable in the first integral to $y = \frac{1}{x}$:

$$
\begin{aligned}
\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} &= \frac{1}{s(s-1)} + \int_{\infty}^1 \left(y^{-1}\right)^{\frac{s}{2}-1}\sqrt{y}\cdot\psi(y)\left(-\frac{dy}{y^2}\right) + \int_1^{\infty} x^{\frac{s}{2}-1}\psi(x)dx \\
&= \frac{1}{s(s-1)} - \int_1^{\infty} \frac{y^{-\frac{s}{2}+\frac{3}{2}}}{y^2}\psi(y)\,(-dy) + \int_1^{\infty} x^{\frac{s}{2}-1}\psi(x)dx \\
&= \frac{1}{s(s-1)} + \int_1^{\infty} y^{-\frac{s}{2}-\frac{1}{2}}\psi(y)dy + \int_1^{\infty} x^{\frac{s}{2}-1}\psi(x)dx \\
&= \frac{1}{s(s-1)} + \int_1^{\infty} \left(x^{-\frac{s}{2}-\frac{1}{2}} + x^{\frac{s}{2}-1}\right)\psi(x)dx \\
&= \frac{1}{s(s-1)} + \int_1^{\infty} \frac{\psi(x)}{x}\left(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right)dx.
\end{aligned}
\tag{26}
$$

It's at this point that Riemann defines the ξ function:

$$\xi(t) := \Gamma\left(\frac{s}{2}+1\right)(s-1)\pi^{-\frac{s}{2}}\zeta(s) \tag{27}$$

where $s = \frac{1}{2} + it$ and $t \in \mathbb{C}$.

Remark 4.1. *This is the original definition found in the paper; it is sometimes considered unintuitive and has mostly been substituted with the definition:*

$$\xi(s) := \frac{s(s-1)}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)$$

for $s \in \mathbb{C}$, which is completely equivalent.
Additionally, it is important to note that equation 21 proves that: $\xi(s) = \xi(1-s)$ (or $\xi(t) = \xi(1-t)$ in Riemann's notation), a more analytical proof of the same property can be found on our site.

What we have proven is that:

$$\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} = \frac{1}{s(s-1)} + \int_1^\infty \psi(x)\left(x^{\frac{s}{2}-1} + x^{-\frac{1+s}{2}}\right)dx$$

$$\Downarrow$$

$$\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\frac{s}{2}(s-1) = \frac{1}{s(s-1)}\cdot\frac{s}{2}(s-1)+\frac{s}{2}(s-1)\int_1^\infty \psi(x)\left(x^{\frac{s}{2}-1} + x^{-\frac{1+s}{2}}\right)dx$$

$$\Downarrow$$

$$\zeta(s)\Gamma\left(\frac{s}{2}+1\right)\pi^{-\frac{s}{2}}(s-1) = \frac{1}{2} + \frac{s}{2}(s-1)\int_1^\infty \psi(x)\left(x^{\frac{s}{2}-1} + x^{-\frac{1+s}{2}}\right)dx$$

$$\Downarrow$$

$$\xi(t) = \frac{1}{2} + \frac{s}{2}(s-1)\int_1^\infty \psi(x)\left(x^{\frac{s}{2}-1} + x^{-\frac{1+s}{2}}\right)dx \tag{28}$$

where we used the known property of the Gamma function: $z\Gamma(z) = \Gamma(z+1)$.

Therefore, remembering that $s = \frac{1}{2} + it$:

$$\begin{aligned}
\xi(t) &= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}+it\right)\left(\frac{1}{2}+it-1\right)\int_1^\infty \psi(x)\left(x^{\left(\frac{1}{2}+it\right)\frac{1}{2}-1} + x^{-\frac{1+\left(\frac{1}{2}+it\right)}{2}}\right)dx \\
&= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}+it\right)\left(-\frac{1}{2}+it\right)\int_1^\infty \psi(x)\left(x^{-\frac{3}{4}+i\frac{t}{2}} + x^{-\frac{3}{4}-i\frac{t}{2}}\right)dx \\
&= \frac{1}{2} + \frac{1}{2}\left(-\frac{1}{4}-t^2\right)\int_1^\infty \psi(x)x^{-\frac{3}{4}}\left(x^{i\frac{t}{2}} + x^{-i\frac{t}{2}}\right)dx \\
&= \frac{1}{2} - \frac{1}{2}\left(t^2+\frac{1}{4}\right)\int_1^\infty \psi(x)x^{-\frac{3}{4}}\left(e^{i\frac{t}{2}\log(x)} + e^{-i\frac{t}{2}\log(x)}\right)dx \\
&= \frac{1}{2} - \frac{1}{2}\left(t^2+\frac{1}{4}\right)\int_1^\infty \psi(x)x^{-\frac{3}{4}}2\cos\left(\frac{t}{2}\log(x)\right)dx.
\end{aligned} \tag{29}$$

We obtain the equation:

$$\xi(t) = \frac{1}{2} - \left(t^2+\frac{1}{4}\right)\int_1^\infty \psi(x)x^{-\frac{3}{4}}\cos\left(\frac{t}{2}\log(x)\right)dx. \tag{30}$$

Riemann proceeds by claiming that the ξ function also satisfies:

$$\xi(t) = 4\int_1^\infty \frac{d}{dx}\left[x^{\frac{3}{2}}\psi'(x)\right]x^{-\frac{1}{4}}\cos\left(\frac{t}{2}\log(x)\right)dx. \tag{31}$$

this is not obvious and requires a few steps, we will add some detail to the passages laid out in [1].

Starting again from 28 we have:

$$\xi(t) = \frac{1}{2} + \frac{s}{2}(s-1)\int_1^\infty \psi(x)\left(x^{\frac{s}{2}-1} + x^{-\frac{1+s}{2}}\right)dx = \frac{1}{2} - \frac{s}{2}(1-s)\int_1^\infty \frac{\psi(x)}{x}\left(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right)dx$$

integrating by parts:

$$= \frac{1}{2} - \frac{s}{2}(1-s)\int_1^\infty \frac{d}{dx}\left\{\psi(x)\left[\frac{x^{\frac{s}{2}}}{\frac{s}{2}} + \frac{x^{\frac{1-s}{2}}}{\frac{1-s}{2}}\right]\right\}dx + \frac{s}{2}(1-s)\int_1^\infty \psi'(x)\left[\frac{x^{\frac{s}{2}}}{\frac{s}{2}} + \frac{x^{\frac{1-s}{2}}}{\frac{1-s}{2}}\right]dx$$

$$= \frac{1}{2} - \frac{s}{2}(1-s)\left|\psi(x)\frac{x^{\frac{s}{2}}}{\frac{s}{2}} + \frac{x^{\frac{1-s}{2}}}{\frac{1-s}{2}}\right|_1^\infty + \frac{s}{2}(1-s)\int_1^\infty \psi'(x)\left[\frac{x^{\frac{s}{2}}}{\frac{s}{2}} + \frac{x^{\frac{1-s}{2}}}{\frac{1-s}{2}}\right]dx$$

$$= \frac{1}{2} + \frac{s}{2}(1-s)\psi(1)\left[\frac{2}{s} + \frac{2}{1-s}\right] + \int_1^\infty \psi'(x)\left[(1-s)x^{\frac{s}{2}} + sx^{\frac{1-s}{2}}\right]dx$$

$$= \frac{1}{2} + \psi(1) + \int_1^\infty \psi'(x)\left[(1-s)x^{\frac{s}{2}} + sx^{\frac{1-s}{2}}\right]dx = \frac{1}{2} + \psi(1) + \int_1^\infty \psi'(x)x^{\frac{1}{2}}\left[(1-s)x^{\frac{s-1}{2}} + sx^{-\frac{s}{2}}\right]dx$$

integrating by parts again:

$$= \frac{1}{2} + \psi(1) + \int_1^\infty \frac{d}{dx}\left[x^{\frac{3}{2}}\psi'(x)(-2x^{\frac{s-1}{2}} - 2x^{-\frac{s}{2}})\right]dx - \int_1^\infty \frac{d}{dx}\left[x^{\frac{3}{2}}\psi'(x)\right](-2x^{\frac{s-1}{2}} - 2x^{-\frac{s}{2}})dx$$

$$= \frac{1}{2} + \psi(1) + \left|x^{\frac{3}{2}}\psi'(x)(-2x^{\frac{s-1}{2}} - 2x^{-\frac{s}{2}})\right|_1^\infty + \int_1^\infty \frac{d}{dx}\left[x^{\frac{3}{2}}\psi'(x)\right](2x^{\frac{s-1}{2}} + 2x^{-\frac{s}{2}})dx$$

$$= \frac{1}{2} + \psi(1) - \psi'(1)(-2-2) + \int_1^\infty \frac{d}{dx}\left[x^{\frac{3}{2}}\psi'(x)\right](2x^{\frac{s-1}{2}} + 2x^{-\frac{s}{2}})dx \tag{32}$$

Notice now that differentiating 24 we have:

$$\frac{d}{dx}2\psi(x) + 1 = \frac{d}{dx}x^{-\frac{1}{2}}\left(2\psi\left(\frac{1}{x}\right) + 1\right)$$

$$\Downarrow$$

$$2\psi'(x) = -\frac{1}{2}x^{-\frac{3}{2}}\left(2\psi\left(\frac{1}{x}\right) + 1\right) - x^{-\frac{1}{2}}\left(2\frac{\psi'\left(\frac{1}{x}\right)}{x^2}\right)$$

and therefore for $x = 1$

$$2\psi'(1) = -\frac{1}{2}(2\psi(1) + 1) - 2\psi'(1)$$

$$\Downarrow$$

$$4\psi'(1) + \psi(1) = -\frac{1}{2}.$$

Hence, utilizing this equality in equation 32:

$$\begin{aligned}
\xi(t) &= \frac{1}{2} + \psi(1) + 4\psi'(1) + \int_1^\infty \frac{d}{dx}\left[x^{\frac{3}{2}}\psi'(x)\right](2x^{\frac{s-1}{2}} + 2x^{-\frac{s}{2}})dx \\
&= \frac{1}{2} - \frac{1}{2} + \int_1^\infty \frac{d}{dx}\left[x^{\frac{3}{2}}\psi'(x)\right](2x^{\frac{s-1}{2}} + 2x^{-\frac{s}{2}})dx \\
&= \int_1^\infty \frac{d}{dx}\left[x^{\frac{3}{2}}\psi'(x)\right]x^{-\frac{1}{4}}(2x^{\frac{s}{2}-\frac{1}{4}} + 2x^{-\frac{s}{2}+\frac{1}{4}})dx \\
&= 2\int_1^\infty \frac{d}{dx}\left[x^{\frac{3}{2}}\psi'(x)\right]x^{-\frac{1}{4}}\left(e^{\left(\frac{s}{2}-\frac{1}{4}\right)\log(x)} + e^{-\left(\frac{s}{2}-\frac{1}{4}\right)\log(x)}\right)dx \\
&\text{fixing again } s = \frac{1}{2} + it \text{ we have:} \\
&= 2\int_1^\infty \frac{d}{dx}\left[x^{\frac{3}{2}}\psi'(x)\right]x^{-\frac{1}{4}}\left(e^{i\left(\frac{t}{2}\right)\log(x)} + e^{-i\left(\frac{t}{2}\right)\log(x)}\right)dx \\
&\Downarrow \\
\xi(t) &= 4\int_1^\infty \frac{d}{dx}\left[x^{\frac{3}{2}}\psi'(x)\right]x^{-\frac{1}{4}}\cos\left(\frac{t}{2}\log(x)\right)dx.
\end{aligned} \tag{33}$$

5 The Riemann Hypothesis and the Number of zeros in the critical strip

The following paragraph is crucial, as it presents Riemann's conjecture, which is now known as the **Riemann Hypothesis.** The author begins by stating that the function $\xi(t)$ is finite for all finite values of t, as can be seen from its equations.

He then notices that, for $Re(s) > 1$, using 1 we have:

$$\log(\zeta(s)) = \log\left(\prod_p (1-p^{-s})^{-1}\right) = \sum_p \log\left((1-p^{-s})^{-1}\right) = -\sum_p \log\left((1-p^{-s})\right) \tag{34}$$

that remains finite; also the same is true for the logarithms of the other factors of $\xi(t)$.

Therefore, if $Re(s) = Re\left(\frac{1}{2}+it\right) > 1$ then $\log(\xi(t))$ is finite, hence $\xi(t) \neq 0$. Given that $Re\left(\frac{1}{2}+it\right) = \frac{1}{2} - Im(t)$, it follows that:

$$Im(t) < -\frac{1}{2} \Rightarrow \xi(t) \neq 0.$$

and also:

$$Im(t) > \frac{1}{2} \Rightarrow -Im(t) < -\frac{1}{2} \Rightarrow Im(1-t) < -\frac{1}{2} \Rightarrow \xi(1-t) \neq 0 \Rightarrow \xi(t) \neq 0$$

as we have noticed in Remark 4.1 that $\xi(t) = \xi(1-t)$. It follows that the function $\xi(t)$ can only vanish if the imaginary part of t lies between $-\frac{1}{2}$ and $\frac{1}{2}$.

Remark 5.1. *This last statement is often expressed in a more modern version:*

The Nontrivial Zeros of the Riemann Zeta Function satisfy $0 \leq Re(s) \leq 1$.

It is now well established that they satisfy $0 < Re(s) < 1$*, but this was unknown at the time.*
Let's see why this is an equivalent statement:
Firstly, looking at the definition of $\xi(t)$ *(27) we can see that* $\xi(t) \neq 0 \Rightarrow \zeta(s) \neq 0$ *except for* $\frac{s}{2} + 1 \in \mathbb{Z}_{\leq 0} \Rightarrow s = -2, -4, -6, \cdots$*, as those are singularities for the Gamma Function.*
Remembering now that $Re(s) = \frac{1}{2} - Im(t)$ *then:*

$$Im(t) > \frac{1}{2},\ Im(t) < -\frac{1}{2} \Rightarrow \xi(t) \neq 0$$

$$\Downarrow$$

$$\begin{cases} Re(s) > 1 \Rightarrow \zeta(s) \neq 0 \\ Re(s) < 0, s \neq -2, -4, -6, \cdots \Rightarrow \zeta(s) \neq 0. \end{cases}$$

In the following section, the author explains briefly one last property of the roots of $\xi(t)$:

Riemann claims that the number of roots of $\xi(t) = 0$, whose real parts lie between 0 and a fixed value T is approximately

$$\frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} \tag{35}$$

At this point, he writes a brief sentence that will change the future of mathematics:

"Man findet nun in der That etwa so viel reelle Wurzeln innerhalb dieser Grenzen, und es ist sehr wahrscheinlich, dass alle Wurzeln reell sind."

"One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real"

This is the original statement of the **Riemann Hypothesis.**

It's equivalent to the modern statement of the Hypothesis:

All Nontrival Zeros of the Riemann Zeta Function lie on the strip $Re(s) = \frac{1}{2}$**.**

To see this simply remember once again that $Re(s) = \frac{1}{2} - Im(t)$ and therefore

$$t \in \mathbb{R} \Rightarrow Im(t) = 0 \Rightarrow Re(s) = \frac{1}{2}.$$

Riemann doesn't provide a rigorous proof of 35 in the paper, it is instead attributed to Von Mangoldt that gave a complete demonstration in 1905, in particular, he proved that the number of zeros of $\zeta(s)$ in the critical range $0 < Re(s) < 1$, $0 < t < T$ is:

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \mathcal{O}(\log(T)). \tag{36}$$

We have also provided an in depth proof in our lesson regarding the Riemann-von Mangoldt formula.

POSITIVE INCREMENT

Thank you!

We hope this lesson has been beneficial in studying this interesting topic.
For more lessons or demonstrations, visit our website.

References

[1] Harold M Edwards. *RiemannÆs zeta function*. Vol. 58. Academic press, 1974.

[2] Bernhard Riemann. "On the number of prime numbers less than a given quantity (ueber die anzahl der primzahlen unter einer gegebenen grösse)". In: *Monatsberichte der Berliner Akademie* (1859).

[3] Bernhard Riemann. "Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse". In: *Ges. Math. Werke und Wissenschaftlicher Nachlaß* 2.145-155 (1859), p. 2.