Equivalences of the Riemann Hypothesis are diverse and spread throughout all of mathematics; however, those typically involve estimates related to arithmetic functions. The one explained in this lesson, instead, is simply an equality:

Theorem 1. *The Riemann Hypothesis is equivalent to the equality:*

$$\int_0^\infty \frac{1-12t^2}{(1+4t^2)^3} dt \int_{\frac{1}{2}}^\infty \log|\zeta(\sigma+it)| d\sigma = \pi \frac{3-\gamma}{32} \tag{1}$$

where γ *is the Euler-Mascheroni constant:* $\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n\right)$.

The following proof uses several results from Davenport's book "Multiplicative Number Theory" [2], along with some from Edward's "Riemann's Zeta Function" [3] and Titchmarsh's and Rodney's "The theory of the Riemann zeta-function" [4]. The entire lesson is a more detailed version of the article where this Theorem was first stated: "On an equality equivalent to the Riemann hypothesis" by V.V. Volchkov [5].

Proof. Start by considering the Riemann ξ function:

$$\xi(s) := (s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}+1\right)\zeta(s). \tag{2}$$

This function also admits a factorization formula:

$$\xi(s) = e^{A+Bs}\prod_\rho \left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}} \tag{3}$$

where the product is over all roots ρ of the ξ function. Remember that the zeros of the ξ function are the non trivial zeros of the ζ function.

Computing the logarithmic derivative of 2 yields:

$$\frac{\xi'(s)}{\xi(s)} = \frac{1}{s-1} - \frac{\log\pi}{2} + \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+1\right)} + \frac{\zeta'(s)}{\zeta(s)}. \tag{4}$$

While, computing the logarithmic derivative of 3 yields:

$$\frac{\xi'(s)}{\xi(s)} = B + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right). \tag{5}$$

Therefore:

$$\begin{aligned}\frac{\zeta'(s)}{\zeta(s)} &= \frac{\xi'(s)}{\xi(s)} - \frac{1}{s-1} + \frac{\log \pi}{2} - \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+1\right)} \\ &= B + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) - \frac{1}{s-1} + \frac{\log \pi}{2} - \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+1\right)}.\end{aligned} \tag{6}$$

Remark 1. *It is interesting to see how this last equation exhibits clearly the pole of $\zeta(s)$ at $s = 1$ and the non trivial zeros $s = \rho$. While the trivial zeros are contained in the term Γ.*

We want to calculate explicitly the term B. Equation 5 implies that:

$$B = \frac{\xi'(0)}{\xi(0)} = -\frac{\xi'(1)}{\xi(1)}$$

due to the functional equation $\xi(s) = \xi(1-s)$.

Using the fact that:

$$-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+1\right)} = \frac{\gamma}{2} + \sum_{n=1}^{\infty}\left(\frac{1}{s+2n} - \frac{1}{2n}\right). \tag{7}$$

Remark 2. *This can be obtained by computing the logarithmic derivative of Weierstrass product for the Gamma Function.*

We have:

$$-\frac{1}{2}\frac{\Gamma'\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} = \frac{\gamma}{2} + \sum_{n=1}^{\infty}\left(\frac{1}{2n+1} - \frac{1}{2n}\right). \tag{8}$$

We will now prove that:

$$-1 + \log 2 - \sum_{n=1}^{\infty}\left(\frac{1}{2n+1} - \frac{1}{2n}\right) = 0 \tag{9}$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{2n+1} - \frac{1}{2n}\right) = -1 + \log 2.$$

We will use Euler's expression of $\log 2$:

$$\log 2 = \sum_{n=0}^{\infty}\frac{1}{(2n+1)(2n+2)}.$$

We have:

$$-1+\log 2 = -1+\sum_{n=0}^{\infty}\frac{1}{(2n+1)(2n+2)} = -1+\sum_{n=0}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n+2}\right)$$
$$= -1+\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n+2}\right) = -\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n+2}\right). \tag{10}$$

Therefore:

$$-1+\log 2-\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n}\right) = -\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n+2}\right)-\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n}\right)$$
$$= -\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n+2}-\frac{1}{2n+1}+\frac{1}{2n}\right)$$
$$= -\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{1}{2n}-\frac{1}{2n+2}\right) = -\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{1}{2n}-\frac{1}{2(n+1)}\right)$$
$$= -\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{n+1-n}{2n(n+1)}\right) = -\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{1}{2n(n+1)}\right) = 0 \tag{11}$$

where in the last equality we used the known telescopic series:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)}\right) = 1$$
$$\Downarrow$$
$$\sum_{n=1}^{\infty}\left(\frac{1}{2n(n+1)}\right) = \frac{1}{2}.$$

In conclusion:

$$-\frac{1}{2}\frac{\Gamma'\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} = \frac{\gamma}{2}-1+\log 2.$$

Hence, substituting this value in equation 4:

$$\frac{\xi'(1)}{\xi(1)} = \lim_{s\to 1}\left[\frac{1}{s-1}+\frac{\zeta'(s)}{\zeta(s)}\right]-\frac{\log\pi}{2}+\frac{1}{2}\frac{\Gamma'\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}$$
$$\Downarrow$$
$$B = \frac{\log\pi}{2}+\frac{\gamma}{2}-1+\log 2-\lim_{s\to 1}\left[\frac{1}{s-1}+\frac{\zeta'(s)}{\zeta(s)}\right]$$
$$\Downarrow$$
$$B = \frac{\log 4\pi}{2}+\frac{\gamma}{2}-1-\lim_{s\to 1}\left[\frac{1}{s-1}+\frac{\zeta'(s)}{\zeta(s)}\right].$$

You can find on our site the Limit of the Logarithmic Derivative:

$$\lim_{s\to 1}\left[\frac{1}{s-1}+\frac{\zeta'(s)}{\zeta(s)}\right]=\gamma.$$

Therefore:

$$B=\frac{\log 4\pi}{2}-\frac{\gamma}{2}-1.$$

We can give another interpretation of B: The series $\sum\rho^{-1}$ converges, provided one groups together the terms from ρ and $\overline{\rho}$. If $\rho=\sigma+it$, then:

$$\frac{1}{\rho}+\frac{1}{\overline{\rho}}=\frac{2\sigma}{\sigma^2+\gamma^2}\leq\frac{2}{|\rho|^2}$$

and we know that $\sum|\rho|^{-2}$ converges.

It follows from the functional equation for the ξ function that:

$$\frac{\xi'(s)}{\xi(s)}=-\frac{\xi'(1-s)}{\xi(1-s)}$$

and therefore, using equation 5:

$$B+\sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)=-B-\sum_{\rho}\left(\frac{1}{1-s-\rho}+\frac{1}{\rho}\right)$$

$$\Downarrow$$

$$2B=\sum_{\rho}\left(-\frac{1}{s-\rho}-\frac{1}{\rho}-\frac{1}{1-s-\rho}-\frac{1}{\rho}\right).$$

It is a known property of the ξ function that if ρ is a zero if and only if $1-\rho$ is a zero, therefore the terms containing $1-s-\rho=-s+(1-\rho)$ and $s-\rho$ are identical and cancel each other.

We are left with:

$$2B=-2\sum_{\rho}\frac{1}{\rho}$$

$$\Downarrow$$

$$B=-\sum_{\rho}\frac{1}{\rho}=-2\sum_{t>0}\frac{\sigma}{\sigma^2+t^2}.$$

In conclusion:

$$\sum_{t>0}\frac{\sigma}{\sigma^2+t^2}=\frac{1}{2}+\frac{\gamma}{4}+\frac{\log 4\pi}{4}.$$

Defining the function f_t as:

$$f_t(\sigma):=\frac{\sigma}{\sigma^2+t^2} \tag{12}$$

We have just proven that:

$$\sum_{t>0} f_t(\sigma) = \frac{\gamma}{4} + \frac{1}{2} + \frac{\log 4\pi}{4}. \tag{13}$$

An important step to prove Theorem 1 is understanding that:

The Riemann Hypothesis is true

$$\Updownarrow$$

$$\sum_{t>0} f_t\left(\frac{1}{2}\right) = \frac{\gamma}{4} + \frac{1}{2} + \frac{\log 4\pi}{4}. \tag{14}$$

This is due to the fact that $f_t(\sigma) > 0$ for all t, σ in the critical line, therefore the sum is strictly increasing. Also, the zeros ρ of the zeta function are symmetrical with respect to the critical line.

Hence, if one supposes true **RH**, the sum over all roots has to coincide with the sum over the roots with $\sigma = \frac{1}{2}$, which implies 14.

Conversely, if 14 is true than there cannot exist a root with $\sigma \neq \frac{1}{2}$, otherwise, by symmetry, there would also be such a root with $t > 0$ and therefore $\sum_{t>0} f_t(\sigma) > \sum_{t>0} f_t\left(\frac{1}{2}\right) = \frac{\gamma}{4} + \frac{1}{2} + \frac{\log 4\pi}{4}$ which is a contradiction.

The conclusion of Theorem 1 follows from a computation of the sum: $\sum_{t>0} f_t\left(\frac{1}{2}\right)$, we will denote it by A.

We have:

$$A = \int_0^\infty f_x\left(\frac{1}{2}\right) dN(x) = \int_0^\infty \frac{dN(x)}{\frac{1}{2} + 2x^2}$$

here $N(x)$ is the number of zeros of $\zeta(s)$ in the region $0 < \sigma \leq 1$, $0 \leq t \leq x$ ($s = \sigma + it$).

Remark 3. *This is an example of a Riemann-Stieltjes integral, for details about this theory we recommend the book "The Stieltjes integral" by G.Convertito and D. Cruz-Uribe [1].*
For those less interested in the complete theory, the wikipedia page on the topic should offer sufficient knowledge.

The Riemann-Stieltjes integral admits integration by parts, hence:

$$\begin{aligned} A &= \int_0^\infty \frac{dN(x)}{\frac{1}{2} + 2x^2} = \left|\frac{N(x)}{\frac{1}{2} + 2x^2}\right|_0^\infty - \int_0^\infty \left(-\frac{16x}{(1 + 4x^2)^2}\right) N(x)\mathrm{d}x \\ &= \int_0^\infty \frac{16x}{(1 + 4x^2)^2} N(x)\mathrm{d}x. \end{aligned} \tag{15}$$

Due to the fact that $N(0) = 0$ and $N(x) = \mathcal{O}(x \log x)$, so $\lim_{x\to\infty} \frac{N(x)}{\frac{1}{2} + 2x^2} = 0$.

Remark 4. *The behavior of the function $N(x)$ for large x has been studied for more than a century. This particular formula can be found in several text, for example [4] or [3]. Also, we have analyzed it thoroughly in our lesson regarding the Riemann-von Mangoldt Formula.*

It is known that:

$$N(x) = 1 - \frac{x\log\pi}{2\pi} + \frac{\mathrm{Im}\left(\log\left(\Gamma\left(\frac{1}{4}+\frac{ix}{2}\right)\right)\right)}{\pi} + S(x)$$

where $S(x) = \frac{(\Delta_L arg\zeta(s))}{\pi}$ is the increment of the argument of $\zeta(s)$ along a polygonal line with vertices at $s = 2$, $s = 2 + ix$,$s = \frac{1}{2} + ix$.

Remark 5. *This can be found again in [4], page 212.*

Hence:

$$\begin{aligned} A &= \int_0^\infty \frac{16x}{\left(1+4x^2\right)^2} N(x)\mathrm{d}x = \int_0^\infty \frac{16x}{\left(1+4x^2\right)^2}\left(1 - \frac{x\log\pi}{2\pi} + \frac{\mathrm{Im}\left(\log\left(\Gamma\left(\frac{1}{4}+\frac{ix}{2}\right)\right)\right)}{\pi} + S(x)\right)\mathrm{d}x \\ &= \int_0^\infty \frac{16x}{\left(1+4x^2\right)^2}\mathrm{d}x - \frac{\log\pi}{2\pi}\int_0^\infty \frac{16x^2}{\left(1+4x^2\right)^2}\mathrm{d}x + \frac{I_2}{\pi} + I_1 \end{aligned} \tag{16}$$

where

$$I_1 = \int_0^\infty S(x)\frac{16x}{\left(1+4x^2\right)^2}\mathrm{d}x, \qquad I_2 = \mathrm{Im}\left(\int_0^\infty \log\left(\Gamma\left(\frac{1}{4}+\frac{ix}{2}\right)\right)\frac{16x}{\left(1+4x^2\right)^2}\mathrm{d}x\right).$$

Computing the first two integrals yields:

$$A = 2 - \frac{\log\pi}{4} + I_1 + \frac{I_2}{\pi}.$$

The integral I_2 can be calculated using integration by parts and the fact that:

$$\frac{\mathrm{d}}{\mathrm{d}x}\log\Gamma\left(\frac{1}{4}+\frac{ix}{2}\right) = \psi\left(\frac{1}{4}+\frac{ix}{2}\right)$$

where ψ is the digamma function. Therefore:

$$\begin{aligned} &\int_0^\infty \log\left(\Gamma\left(\frac{1}{4}+\frac{ix}{2}\right)\right)\frac{16x}{\left(1+4x^2\right)^2}\mathrm{d}x \\ &= \left|-\left(\frac{2}{4x^2+1}\right)\log\left(\Gamma\left(\frac{1}{4}+\frac{ix}{2}\right)\right)\right|_0^\infty + \int_0^\infty \left(\frac{2}{4x^2+1}\right)\psi\left(\frac{1}{4}+\frac{ix}{2}\right)\mathrm{d}x \\ &= -2\log\left(\Gamma\left(\frac{1}{4}\right)\right) + \int_0^\infty \left(\frac{2}{4x^2+1}\right)\psi\left(\frac{1}{4}+\frac{ix}{2}\right)\mathrm{d}x \end{aligned} \tag{17}$$

$$\Downarrow$$

$$\begin{aligned} I_2 &= \mathrm{Im}\left(\int_0^\infty \log\left(\Gamma\left(\frac{1}{4}+\frac{ix}{2}\right)\right)\frac{16x}{\left(1+4x^2\right)^2}\mathrm{d}x\right) \\ &= \mathrm{Im}\left(\int_0^\infty \left(\frac{2}{4x^2+1}\right)\psi\left(\frac{1}{4}+\frac{ix}{2}\right)\mathrm{d}x\right) = -\frac{\gamma}{4}\pi - \frac{\pi}{2}\log 2. \end{aligned} \tag{18}$$

$$\Downarrow$$

$$A = 2 - \frac{\log \pi}{4} + I_1 - \frac{\gamma}{4} - \frac{\log 2}{2}.$$

We can also evaluate I_1 using integration by parts and the estimate:

$$S_1(x) = \int_0^x S(t)\mathrm{d}t = \mathcal{O}(\log x).$$

Remark 6. *This can be found in [4], page 222.*

We have:

$$I_1 = \int_0^\infty S(x)\frac{16x}{(1+4x^2)^2}\mathrm{d}x = \left| S_1(x)\frac{16x}{(1+4x^2)^2}\right|_0^\infty - \int_0^\infty S_1(x)\frac{16(1-12x^2)}{(4x^2+1)^3}\mathrm{d}x$$
$$= -\int_0^\infty S_1(x)\frac{16(1-12x^2)}{(4x^2+1)^3}\mathrm{d}x. \tag{19}$$

Using now the fact that ([4], page 221):

$$S_1(x) = \frac{1}{\pi}\int_{\frac{1}{2}}^\infty \log|\zeta(\sigma+it)|\mathrm{d}\sigma$$

we conclude:

$$\sum_{t>0} f_t\left(\frac{1}{2}\right) = \frac{\gamma}{4} + \frac{1}{2} + \frac{\log 4\pi}{4}. \tag{20}$$

$$\Updownarrow$$

$$2 - \frac{\log \pi}{4} - \int_0^\infty S_1(x)\frac{16(1-12x^2)}{(4x^2+1)^3}\mathrm{d}x - \frac{\gamma}{4} - \frac{\log 2}{2} = \frac{\gamma}{4} + \frac{1}{2} + \frac{\log 4\pi}{4}$$

$$\Updownarrow$$

$$\int_0^\infty S_1(x)\frac{16(1-12x^2)}{(4x^2+1)^3}\mathrm{d}x = 2 - \frac{\log \pi}{4} - \frac{\gamma}{4} - \frac{\gamma}{4} - \frac{1}{2} - \frac{\log 4\pi}{4} - \frac{\log 2}{2}$$

$$\Updownarrow$$

$$\int_0^\infty S_1(x)\frac{16(1-12x^2)}{(4x^2+1)^3}\mathrm{d}x = -\frac{\gamma}{2} + \frac{3}{2} - \frac{\log 4\pi - 2\log 2 - \log \pi}{4}$$

$$\Updownarrow$$

$$\int_0^\infty S_1(x)\frac{16(1-12x^2)}{(4x^2+1)^3}\mathrm{d}x = \frac{3-\gamma}{2}$$

$$\Updownarrow$$

$$\frac{1}{\pi}\int_{\frac{1}{2}}^\infty \log|\zeta(\sigma+it)|\mathrm{d}\sigma \int_0^\infty \frac{(1-12x^2)}{(4x^2+1)^3}\mathrm{d}x = \frac{3-\gamma}{32}$$

$$\Updownarrow$$

$$\int_{\frac{1}{2}}^\infty \log|\zeta(\sigma+it)|\mathrm{d}\sigma \int_0^\infty \frac{(1-12x^2)}{(4x^2+1)^3}\mathrm{d}x = \pi\frac{3-\gamma}{32}$$

which proves 1. □

POSITIVE INCREMENT

Thank you!

We hope this lesson has been beneficial in studying this interesting topic.
For more lessons or demonstrations, visit our website.

References

[1] Gregory Convertito and David Cruz-Uribe. *The Stieltjes Integral*. Chapman and Hall/CRC, 2023.

[2] Harold Davenport. *Multiplicative number theory*. Vol. 74. Springer Science & Business Media, 2013.

[3] Harold M Edwards. *RiemannÆs zeta function*. Vol. 58. Academic press, 1974.

[4] Edward Charles Titchmarsh and David Rodney Heath-Brown. *The theory of the Riemann zeta-function*. Oxford university press, 1986.

[5] VV Volchkov. "On an equality equivalent to the Riemann hypothesis". In: *Ukrainian Mathematical Journal* 47.3 (1995), pp. 491–493.