Today we will focus on the article "Sur les zéros de la fonction Zeta de Riemann" by Godfrey Harold Hardy [2].
It's a brief article, but the main result is considered the first step made toward solving the Riemann Hypothesis:

Theorem 1. *There is an infinite number of $t \in \mathbb{R}$ such that:*

$$\zeta\left(\frac{1}{2} + it\right) = 0.$$

Proof. Begin by considering the formula:

$$e^{-y} = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \Gamma(u) y^{-u} du \tag{1}$$

true for $Re(y) > 0$ and $k > 0$.

This is often refereed to as the **Cahen-Mellin Integral**, it is obtained using the Mellin transformation $\mathcal{M}$. We will go briefly through the steps necessary to arrive to the equality, the details can be found in [1].

Given a function $f(y)$ defined on the positive real axis $0 < y < \infty$, the **Mellin transformation** $\mathcal{M}$ is the operation mapping f in the function F defined in the complex plane as:

$$\mathcal{M}[f; u] \equiv F(u) := \int_0^\infty f(y) y^{u-1} dy$$

$F(u)$ is called the **Mellin transform** of f.

By definition, for $f(y) = e^{-py}$, $p > 0$, we have:

$$\mathcal{M}[e^{-py}; u] = \int_0^\infty e^{-py} y^{u-1} dy = \int_0^\infty e^{-y} \left(\frac{y}{p}\right)^{u-1} \frac{dy}{p} = p^{-u} \int_0^\infty e^{-y} y^{u-1} dy = p^{-u}\Gamma(u)$$

for $Re(u) > 0$.

Using Fourier's inversion theorem, one can find a direct way to invert Mellin's transformation in:

$$f(y) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} F(u)y^{-u}du.$$

Therefore, using the fact that $F(u) = \mathcal{M}[e^{-py}; u] = p^{-u}\Gamma(u)$, we obtain:

$$e^{-py} = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} y^{-u}p^{-u}\Gamma(u)du. \tag{2}$$

Remark 1. *This is a more general form of equation 1, it is never mentioned in the article but, as we will see, it is necessary to proceed with the proof.*

Substitute $p = n^2$, $n \neq 0$, in equation 2 to obtain:

$$e^{-n^2 y} = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} y^{-u}n^{-2u}\Gamma(u)du$$

$$\Downarrow$$

$$2e^{-n^2 y} = \frac{1}{\pi i}\int_{k-i\infty}^{k+i\infty} y^{-u}n^{-2u}\Gamma(u)du$$

$$\Downarrow$$

$$1 + 2e^{-n^2 y} = 1 + \frac{1}{\pi i}\int_{k-i\infty}^{k+i\infty} y^{-u}n^{-2u}\Gamma(u)du$$

$$\Downarrow$$

$$1 + 2\sum_{n=1}^{\infty} e^{-n^2 y} = 1 + \sum_{n=1}^{\infty}\frac{1}{\pi i}\int_{k-i\infty}^{k+i\infty} y^{-u}n^{-2u}\Gamma(u)du$$

$$\Downarrow$$

$$1 + 2\sum_{n=1}^{\infty} e^{-n^2 y} = 1 + \frac{1}{\pi i}\int_{k-i\infty}^{k+i\infty} y^{-u}\sum_{n=1}^{\infty} n^{-2u}\Gamma(u)du$$

$$\Downarrow$$

$$1 + 2\sum_{n=1}^{\infty} e^{-n^2 y} = 1 + \frac{1}{\pi i}\int_{k-i\infty}^{k+i\infty} y^{-u}\zeta(2u)\Gamma(u)du. \tag{3}$$

Notice that we are allowed to change the order of integration and series due to absolute convergence.

The above formula is only true for $k > \frac{1}{2}$, as the function $\zeta(2s)$ has a pole at $s = \frac{1}{2}$, we can therefore choose to move the integration contour left, if we remember to add, using the residue Theorem, $2\pi i$ times the residue at $s = \frac{1}{2}$, which is $\frac{1}{2\pi i}\sqrt{\frac{\pi}{y}}$, that is to say:

$$1 + \frac{1}{\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} y^{-u}\zeta(2u)\Gamma(u)du = 1 + \sqrt{\frac{\pi}{y}} + \frac{1}{\pi i}\int_{\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} y^{-u}\zeta(2u)\Gamma(u)du. \tag{4}$$

Remember now the definition of Riemann's ξ function:

$$\xi(s) := \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{s}{2}\right)$$

and of the Ξ function:

$$\Xi(t) := \xi\left(\frac{1}{2} + it\right).$$

Using these definitions:

$$\Xi(2t) = \xi\left(\frac{1}{2} + 2it\right) = \xi\left(2\left(\frac{1}{4} + it\right)\right) = \left(\frac{1}{4} + it\right)\left(\frac{1}{2} + 2it - 1\right)\pi^{-\frac{1}{4}-it}\zeta\left(\frac{1}{2} + 2it\right)\Gamma\left(\frac{1}{4} + it\right)$$

$$\Downarrow$$

$$\zeta\left(\frac{1}{2} + 2it\right)\Gamma\left(\frac{1}{4} + it\right) = \frac{\Xi(2t)\pi^{\frac{1}{4}+it}}{\left(\frac{1}{4} + it\right)\left(\frac{1}{2} + 2it - 1\right)}. \tag{5}$$

Now, changing variable to $u = \frac{1}{4} + it$, the integral in equation 4 becomes:

$$\frac{1}{\pi i}\int_{\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} y^{-u}\zeta(2u)\Gamma(u)du = \frac{1}{\pi i}\int_{-\infty}^{+\infty}\left(\frac{1}{y}\right)^{\frac{1}{4}+it}\zeta\left(\frac{1}{2} + 2it\right)\Gamma\left(\frac{1}{4} + it\right)idt \tag{6}$$

combining with equation 5 we find:

$$\begin{aligned}
&= \frac{1}{\pi}\int_{-\infty}^{+\infty}\left(\frac{\pi}{y}\right)^{\frac{1}{4}+it}\frac{\Xi(2t)}{\left(\frac{1}{4} + it\right)\left(-\frac{1}{2} + 2it\right)}dt = \frac{1}{\pi}\int_{-\infty}^{+\infty}\left(\frac{\pi}{y}\right)^{\frac{1}{4}+it}\frac{\Xi(2t)}{-\frac{1}{2}\left(\frac{1}{4} + 4t^2\right)}dt\\
&= -\frac{2}{\pi}\int_{-\infty}^{+\infty}\left(\frac{\pi}{y}\right)^{\frac{1}{4}+it}\frac{\Xi(2t)}{\frac{1}{4} + 4t^2}dt.
\end{aligned} \tag{7}$$

Therefore, equation 3 can be written as:

$$1 + 2\sum_{n=1}^{\infty} e^{-n^2 y} = 1 + \sqrt{\frac{\pi}{y}} - \frac{2}{\pi}\int_{-\infty}^{+\infty}\left(\frac{\pi}{y}\right)^{\frac{1}{4}+it}\frac{\Xi(2t)}{\frac{1}{4} + 4t^2}dt. \tag{8}$$

Notice that Ξ is an even function:

$$\Xi(t) = \xi\left(\frac{1}{2} + it\right) = \xi\left(1 - \frac{1}{2} - it\right) = \xi\left(\frac{1}{2} - it\right) = \Xi(-t)$$

where we used the know property of the ξ function: $\xi(s) = \xi(1 - s)$ a rigorous proof of this can be found on our site.

Using this property, we can rewrite the integral in 8 as:

$$\begin{aligned}
&\frac{2}{\pi}\int_{-\infty}^{+\infty}\left(\frac{\pi}{y}\right)^{\frac{1}{4}+it}\frac{\Xi(2t)}{\frac{1}{4} + 4t^2}dt = \frac{2}{\pi}\int_{0}^{+\infty}\left(\frac{\pi}{y}\right)^{\frac{1}{4}+it}\frac{\Xi(2t)}{\frac{1}{4} + 4t^2}dt + \frac{2}{\pi}\int_{-\infty}^{0}\left(\frac{\pi}{y}\right)^{\frac{1}{4}+it}\frac{\Xi(2t)}{\frac{1}{4} + 4t^2}dt\\
&= \frac{2}{\pi}\int_{0}^{+\infty}\left(\frac{\pi}{y}\right)^{\frac{1}{4}+it}\frac{\Xi(2t)}{\frac{1}{4} + 4t^2}dt + \frac{2}{\pi}\int_{+\infty}^{0}\left(\frac{\pi}{y}\right)^{\frac{1}{4}-it}\frac{\Xi(-2t)}{\frac{1}{4} + 4(-t)^2}(-dt)\\
&= \frac{2}{\pi}\int_{0}^{+\infty}\left(\frac{\pi}{y}\right)^{\frac{1}{4}+it}\frac{\Xi(2t)}{\frac{1}{4} + 4t^2}dt + \frac{2}{\pi}\int_{0}^{+\infty}\left(\frac{\pi}{y}\right)^{\frac{1}{4}-it}\frac{\Xi(2t)}{\frac{1}{4} + 4t^2}dt\\
&= \frac{2}{\pi}\int_{0}^{+\infty}\frac{\Xi(2t)}{\frac{1}{4} + 4t^2}\left(\left(\frac{\pi}{y}\right)^{\frac{1}{4}+it} + \left(\frac{\pi}{y}\right)^{\frac{1}{4}-it}\right)dt.
\end{aligned} \tag{9}$$

Replacing y with $y = \pi e^{i\alpha}$, $-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi$, we have:

$$1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi e^{i\alpha}} = 1 + \sqrt{\frac{\pi}{\pi e^{i\alpha}}} - \frac{2}{\pi}\int_0^{+\infty} \frac{\Xi(2t)}{\frac{1}{4}+4t^2}\left(\left(\frac{\pi}{\pi e^{i\alpha}}\right)^{\frac{1}{4}+it} + \left(\frac{\pi}{\pi e^{i\alpha}}\right)^{\frac{1}{4}-it}\right)dt$$

$$\Downarrow$$

$$1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi e^{i\alpha}} = 1 + e^{-\frac{1}{2}i\alpha} - \frac{2}{\pi}\int_0^{+\infty} \frac{\Xi(2t)}{\frac{1}{4}+4t^2}\left(e^{-\frac{1}{4}i\alpha}\left(e^{\alpha t} + e^{-\alpha t}\right)\right)dt$$

$$\Downarrow$$

$$1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi e^{i\alpha}} = 1 + e^{-\frac{1}{2}i\alpha} - \frac{2e^{-\frac{1}{4}i\alpha}}{\pi}\int_0^{+\infty} \frac{\left(e^{\alpha t} + e^{-\alpha t}\right)\Xi(2t)}{\frac{1}{4}+4t^2}dt$$

$$\Downarrow$$

$$1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi e^{i\alpha}} - 1 - e^{-\frac{1}{2}i\alpha} = -\frac{2e^{-\frac{1}{4}i\alpha}}{\pi}\int_0^{+\infty} \frac{\left(e^{\alpha t} + e^{-\alpha t}\right)\Xi(2t)}{\frac{1}{4}+4t^2}dt.$$

Multiply both sides by $-\frac{\pi}{2}e^{\frac{1}{4}i\alpha}$ to obtain

$$-\frac{\pi}{2}e^{\frac{1}{4}i\alpha}\left(1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi e^{i\alpha}}\right) + \frac{\pi}{2}e^{\frac{1}{4}i\alpha} + \frac{\pi}{2}e^{\frac{1}{4}i\alpha-\frac{1}{2}i\alpha} = \int_0^{+\infty} \frac{\left(e^{\alpha t} + e^{-\alpha t}\right)\Xi(2t)}{\frac{1}{4}+4t^2}dt$$

$$\Downarrow$$

$$-\frac{\pi}{2}e^{\frac{1}{4}i\alpha}\left(1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi e^{i\alpha}}\right) + \frac{\pi}{2}\left(e^{\frac{1}{4}i\alpha} + e^{-\frac{1}{4}i\alpha}\right) = \int_0^{+\infty} \frac{\left(e^{\alpha t} + e^{-\alpha t}\right)\Xi(2t)}{\frac{1}{4}+4t^2}dt. \tag{10}$$

In conclusion, we have:

$$\int_0^{\infty} \frac{(e^{\alpha t} + e^{-\alpha t})\Xi(2t)}{\frac{1}{4}+4t^2}dt = -\frac{\pi}{2}e^{\frac{1}{4}i\alpha}(1 + 2\psi(e^{i\alpha})) + \pi\cos\left(\frac{1}{4}\alpha\right) \tag{11}$$

where

$$\psi(x) = \sum_{1}^{+\infty} e^{-n^2\pi x}.$$

Remark 2. *In the original article Hardy writes $F(q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}$, instead of $\psi(x)$, where $q = e^{-\pi e^{i\alpha}}$.*
*He also notices that $F(q) = \theta_3(0, \tau)$; the function θ_3 is defined as $\theta_3(z, q) = \sum_{-\infty}^{+\infty} q^{n^2} e^{2\pi i z}$ and is an example of the so called **Jacobi Theta functions**; key functions in many topics of mathematics.*
We are using a different notation to match the one used in [3], given that the following part of the demonstration comes mainly from this other source.

The key formula used by Hardy in the proof of Theorem 1 is obtained from equation 11 differentiating both sides $2p$ times with respect to α.

Note that, provided $\alpha < \frac{1}{2}\pi$, we can differentiate the above integral with respect to α any number of times since $\zeta\left(\frac{1}{2} + it\right) = \mathcal{O}(t^A)$, $\Xi(t) = \mathcal{O}(t^A e^{-\frac{1}{2}\pi t})$.

We therefore have:

$$\int_0^\infty \frac{t^{2p}(e^{\alpha t}+e^{-\alpha t})\Xi(2t)}{\frac{1}{4}+4t^2}dt = \frac{(-1)^p\pi}{4^{2p}}\cos\left(\frac{1}{4}\alpha\right)-\left(\frac{d}{d\alpha}\right)^{2p}\left[\frac{\pi}{2}e^{\frac{1}{4}i\alpha}(1+2\psi(e^{i\alpha}))\right]. \tag{12}$$

The next step in the proof is to consider $\alpha \to \frac{1}{2}\pi$ and prove that the last term of equation 12 tends to 0, for all possible choices of p. Hardy does not prove this in detail, the demonstration that follows is an adaptation of the one in [3]:

For $\alpha \to \frac{1}{2}\pi$, $e^{i\alpha} \to e^{i\frac{1}{2}\pi} = i$, therefore we have to focus on $\psi(x)$ when $x \to i$.

It is a known property of the function $\psi(x)$ that:

$$2\psi(x)+1 = \frac{1}{\sqrt{x}}\left[2\psi\left(\frac{1}{x}\right)+1\right]. \tag{13}$$

This equation comes up often while studying Analytic Number Theory, you can find a demonstration on our site; (mind that there the function is denoted $\omega(x)$).

Equation 13 implies:

$$\psi(x) = \frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right)+\frac{1}{2}\frac{1}{\sqrt{x}}-\frac{1}{2} \tag{14}$$

therefore:

$$\psi(i+\delta) = \sum_{n=1}^{\infty} e^{-n^2\pi(i+\delta)} = \sum_{n=1}^{\infty} e^{-n^2\pi i}e^{-n^2\pi\delta} = \sum_{n=1}^{\infty}(-1)^{-n^2}e^{-n^2\pi\delta} = \sum_{n=1}^{\infty}(-1)^n e^{-n^2\pi\delta}$$

where in the last equality we used the fact that n^2 and n have the same parity.

We can write:

$$\begin{aligned}
\sum_{n=1}^{\infty}(-1)^n e^{-n^2\pi\delta} &= \sum_{n\in\mathbb{Z}_{>0}, n \text{ even}}^{\infty} e^{-n^2\pi\delta} - \sum_{n\in\mathbb{Z}_{>0}, n \text{ odd}}^{\infty} e^{-n^2\pi\delta} \\
&= \sum_{n=1}^{\infty} e^{-(2n)^2\pi\delta} - \left(\sum_{n=1}^{\infty} e^{-n^2\pi\delta} - \sum_{n\in\mathbb{Z}_{>0}, n \text{ even}}^{\infty} e^{-n^2\pi\delta}\right) \\
&= \sum_{n=1}^{\infty} e^{-n^2\pi 4\delta} - \left(\psi(\delta) - \sum_{n=1}^{\infty} e^{-(2n)^2\pi\delta}\right) \\
&= \psi(4\delta) - \psi(\delta) + \sum_{n=1}^{\infty} e^{-n^2\pi 4\delta} \\
&= 2\psi(4\delta) - \psi(\delta).
\end{aligned} \tag{15}$$

Using now equation 14 we have:

$$
\begin{aligned}
2\psi(4\delta)-\psi(\delta) &= \frac{2}{\sqrt{4\delta}}\psi\left(\frac{1}{4\delta}\right)+\frac{1}{\sqrt{4\delta}}-1-\frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{\delta}\right)-\frac{1}{2\sqrt{\delta}}+\frac{1}{2}\\
&= \frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{4\delta}\right)-\frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{\delta}\right)-\frac{1}{2}.
\end{aligned} \tag{16}
$$

From this last equality, one sees that $\psi(i+\delta)\to 0$ for $\delta\to 0$, or in other words that $\psi(x)$ and all its derivatives tend to 0 for $x\to i$ along any route in an angle $|\arg(x-i)|<\frac{1}{2}\pi$.

Therefore, for $\alpha\to\frac{1}{2}\pi$, equation 12 tends to $\frac{(-1)^p\pi}{4^{2p}}\cos\left(\frac{1}{8}\pi\right)$:

$$
\int_0^\infty \frac{t^{2p}(e^{\frac{1}{2}\pi t}+e^{-\frac{1}{2}\pi t})\Xi(2t)}{\frac{1}{4}+4t^2}dt=\frac{(-1)^p\pi}{4^{2p}}\cos\left(\frac{1}{8}\pi\right). \tag{17}
$$

The last step of the proof is the key one, Hardy supposes that we can find a $T>1$ such that the function $\Xi(t)$ never changes its sign once we go over this value and this assumption brings us to a contradiction.
The function $\Xi(t)$ therefore never stops going from positive to negative, which proves that $\Xi(t)=\zeta\left(\frac{1}{2}+it\right)$ has infinitely many zeros.

Suppose that, for every $t>T>1$, $\Xi(t)$ maintains one sign, say positive.

Consider now equation 17 with p odd:

$$
\int_0^\infty \frac{t^{2p}(e^{\frac{1}{2}\pi t}+e^{-\frac{1}{2}\pi t})\Xi(2t)}{\frac{1}{4}+4t^2}dt=\int_0^T \frac{t^{2p}(e^{\frac{1}{2}\pi t}+e^{-\frac{1}{2}\pi t})\Xi(2t)}{\frac{1}{4}+4t^2}dt+\int_T^\infty \frac{t^{2p}(e^{\frac{1}{2}\pi t}+e^{-\frac{1}{2}\pi t})\Xi(2t)}{\frac{1}{4}+4t^2}dt
$$

$$\Downarrow$$

$$
\frac{(-1)^p\pi}{4^{2p}}\cos\left(\frac{1}{8}\pi\right)-\int_0^T \frac{t^{2p}(e^{\frac{1}{2}\pi t}+e^{-\frac{1}{2}\pi t})\Xi(2t)}{\frac{1}{4}+4t^2}dt=\int_T^\infty \frac{t^{2p}(e^{\frac{1}{2}\pi t}+e^{-\frac{1}{2}\pi t})\Xi(2t)}{\frac{1}{4}+4t^2}dt
$$

p is odd, therefore $\frac{(-1)^p\pi}{4^{2p}}\cos\left(\frac{1}{8}\pi\right)<0$ and we have:

$$
\int_T^\infty \frac{t^{2p}(e^{\frac{1}{2}\pi t}+e^{-\frac{1}{2}\pi t})\Xi(2t)}{\frac{1}{4}+4t^2}dt<-\int_0^T \frac{t^{2p}(e^{\frac{1}{2}\pi t}+e^{-\frac{1}{2}\pi t})\Xi(2t)}{\frac{1}{4}+4t^2}dt<KT^{2p} \tag{18}
$$

where $K=-\int_0^T \frac{(e^{\frac{1}{2}\pi t}+e^{-\frac{1}{2}\pi t})\Xi(2t)}{\frac{1}{4}+4t^2}dt$ does not depend on p.

This gives us a contradiction: By our assumption, $\Xi(2t)>\delta$ for some positive δ for every $t\in[2T,2T+1]$, therefore:

$$
\int_T^\infty \frac{t^{2p}(e^{\frac{1}{2}\pi t}+e^{-\frac{1}{2}\pi t})\Xi(2t)}{\frac{1}{4}+4t^2}dt>\int_{2T}^{2T+1} \frac{t^{2p}(e^{\frac{1}{2}\pi t}+e^{-\frac{1}{2}\pi t})\Xi(2t)}{\frac{1}{4}+4t^2}dt>\delta(2T)^{2p}K_1 \tag{19}
$$

where this time $K_1 = \int_{2T}^{2T+1} \frac{(e^{\frac{1}{2}\pi t}+e^{-\frac{1}{2}\pi t})}{\frac{1}{4}+4t^2} dt$ is positive and still does not depend on p.

Now using 18 and 19 we have:

$$KT^{2p} > \delta(2T)^{2p} K_1$$
$$\Downarrow$$
$$K > \delta 2^{2p} K_1 \tag{20}$$

but this should be true for any odd integer p, while, if $K < 0$ it is obviously false and for $K > 0$ is false for any $p > \frac{\log_2\left(\frac{K}{K_1 \delta}\right)}{2}$.

$\square$

POSITIVE INCREMENT

Thank you!

We hope this lesson has been beneficial in studying this interesting topic.
For more lessons or demonstrations, visit our website.

References

[1] Jacqueline Bertrand, Pierre Bertrand, and Jean-Philippe Ovarlez. "The mellin transform". In: *The transforms and applications handbook* (1995).

[2] Godfrey Harold Hardy. "Sur les zéros de la fonction ζ (s) de Riemann". In: *CR Acad. Sci. Paris* 158.1914 (1914), p. 1012.

[3] Edward Charles Titchmarsh and David Rodney Heath-Brown. *The theory of the Riemann zeta-function*. Oxford university press, 1986.