**Theorem 1.** *Let $G(s)$ be an integral function of finite order, $P(s)$ a polynomial and $f(s) = \frac{G(s)}{P(s)}$. Let:*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{1}$$

*be absolutely convergent for* $\mathrm{Re}(s) > 1$ *and*

$$f(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} = g(1-s)\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\pi^{-\frac{(1-s)}{2}} \tag{2}$$

*where*

$$g(1-s) = \sum_{n=1}^{\infty} \frac{b_n}{n^{1-s}}$$

*the series being absolutely convergent for* $\mathrm{Re}(s) < -\alpha < 0$.

*Then $f(s) = C\zeta(s)$ for some constant $C$.*

**Remark 1.** *Most of the following demonstration comes from [1], we added a few details to make it clearer.*

*Proof.* Define $\phi(x)$ as:

$$\phi(x) := \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} f(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}x^{-\frac{s}{2}}\mathrm{d}s.$$

Then, using hypothesis 1, we have, for $x > 0$:

$$\begin{aligned}\phi(x) &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} f(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}x^{-\frac{s}{2}}\mathrm{d}s = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\sum_{n=1}^{\infty}\frac{a_n}{n^s}\Gamma\left(\frac{s}{2}\right)(\pi x)^{-\frac{s}{2}}\mathrm{d}s \\ &= \sum_{n=1}^{\infty}\frac{a_n}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma\left(\frac{s}{2}\right)(\pi n^2 x)^{-\frac{s}{2}}\mathrm{d}s.\end{aligned} \tag{3}$$

This last integral can be computed using the **Mellin Transform.**

Recall that the Mellin Transform of a function $f(x)$ is defined as:

$$F(s) = \mathcal{M}\{f(x); s\} = \int_0^\infty f(x)x^{s-1}\mathrm{d}x.$$

While the inverse Mellin Transform is:

$$f(x) = \mathcal{M}^{-1}\{F(s); x\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)x^{-s}\mathrm{d}s$$

where $c$ is a real number chosen so that the integral on the right converges.

If we define $y = \pi n^2 x$, the integral in equation 3 becomes:

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma\left(\frac{s}{2}\right) y^{-\frac{s}{2}}\mathrm{d}s.$$

Change variable to $t = \frac{s}{2}$ to obtain:

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma\left(\frac{s}{2}\right) y^{-\frac{s}{2}}\mathrm{d}s = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \Gamma\left(t\right) y^{-t} 2\mathrm{d}t = \frac{2}{2\pi i}\int_{1-i\infty}^{1+i\infty} \Gamma\left(t\right) y^{-t}\mathrm{d}t.$$

Recognize that $\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \Gamma\left(t\right) y^{-t}\mathrm{d}t$ is the inverse Mellin Transform of $\Gamma(t)$ evaluated at $y$.

The definition of the Gamma Function is:

$$\Gamma(s) := \int_0^\infty e^{-x}x^{s-1}\mathrm{d}t$$

which one notices is exactly the formula for the Mellin Transform of $\Gamma(s)$. Therefore the inverse Mellin transform of $\Gamma(s)$ is $e^{-x}$:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)x^{-s}\mathrm{d}s = e^{-x}$$

$$\Downarrow$$

$$\phi(x) = \sum_{n=1}^{\infty} \frac{2a_n}{2\pi i}\int_{1-i\infty}^{1+i\infty} \Gamma\left(t\right) y^{-t}\mathrm{d}s = 2\sum_{n=1}^{\infty} a_n e^{-y} = 2\sum_{n=1}^{\infty} a_n e^{-\pi n^2 x}. \tag{4}$$

On the other hand, the functional equation 2 implies:

$$\phi(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} g(1-s)\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\pi^{-\frac{(1-s)}{2}}x^{-\frac{s}{2}}\mathrm{d}s.$$

We want now to move the contour of integration from the vertical line $\mathrm{Re}(s) = 2$ to the vertical line $\mathrm{Re}(s) = -1-\alpha$. To do this we need to ensure that the integrand does not grow too rapidly as $|\mathrm{Im}(s)| \to \infty$.

We know that, by hypothesis, $f(s)$ is bounded on $\mathrm{Re}(s) = 2$ and $g(1-s)$ is bounded on $\mathrm{Re}(s) = -1-\alpha$.
Since:

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)} = \mathcal{O}\left(|t|^{\mathrm{Re}(s)-\frac{1}{2}}\right)$$

it follows that $g(1-s) = \mathcal{O}\left(|t|^{\frac{3}{2}}\right)$ on $\mathrm{Re}(s) = 2$.

We can therefore apply Cauchy's Theorem and move the contour of integration by adding the residues of the poles present in the new region:

$$\phi(x) = \frac{1}{2\pi i}\int_{-\alpha-1-i\infty}^{-\alpha-1+i\infty} g(1-s)\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\pi^{-\frac{(1-s)}{2}}x^{-\frac{s}{2}}\mathrm{d}s + \sum_{\upsilon=1}^{m} R_\upsilon$$

where $R_1, \cdots, R_m$ are the residues at the poles $s_1, \cdots s_m$. Hence:

$$\sum_{\upsilon=1}^{m} R_\upsilon = \sum_{\upsilon=1}^{m} x^{-\frac{s_\upsilon}{2}}\mathcal{P}_\upsilon(\log x) = \mathcal{P}(x)$$

where the $\mathcal{P}_\upsilon(\log x)$ are polynomials in $\log x$, coming from the derivatives in $\mathrm{d}s$ of $x^{-\frac{s}{2}}$.

Thus:

$$\begin{aligned}
\phi(x) &= \frac{1}{2\pi i}\int_{-\alpha-i-\infty}^{-\alpha-1+i\infty} g(1-s)\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\pi^{-\frac{(1-s)}{2}}x^{-\frac{s}{2}}\mathrm{d}s + \sum_{\upsilon=1}^{m} R_\upsilon \\
&= \frac{1}{2\pi i}\int_{-\alpha-i-\infty}^{-\alpha-1+i\infty} \sum_{n=1}^{\infty}\frac{b_n}{n^{1-s}}\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\left(\frac{\pi}{x}\right)^{-\frac{(1-s)}{2}}x^{-\frac{1}{2}}\mathrm{d}s + \mathcal{P}(x) \qquad (5)\\
&= \frac{1}{\sqrt{x}}\sum_{n=1}^{\infty}\frac{b_n}{2\pi i}\int_{-\alpha-i-\infty}^{-\alpha-1+i\infty}\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\left(\frac{\pi n^2}{x}\right)^{-\frac{(1-s)}{2}}\mathrm{d}s + \mathcal{P}(x).
\end{aligned}$$

Using again the Mellin Transform, in almost an identical way this last integral can be written as:

$$\frac{1}{2\pi i}\int_{-\alpha-i-\infty}^{-\alpha-1+i\infty}\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\left(\frac{\pi n^2}{x}\right)^{-\frac{(1-s)}{2}}\mathrm{d}s = 2e^{-\frac{\pi n^2}{x}}.$$

Therefore:

$$\phi(x) = \frac{2}{\sqrt{x}}\sum_{n=1}^{\infty} b_n e^{-\frac{\pi n^2}{x}} + \mathcal{P}(x)$$

Hence, using equation 4, we have:

$$\sum_{n=1}^{\infty} a_n e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}\sum_{n=1}^{\infty} b_n e^{-\frac{\pi n^2}{x}} + \frac{\mathcal{P}(x)}{2}.$$

Multiplying both sides by $e^{-\pi t^2 x}$ with $t > 0$ and integrating over $(0, \infty)$ in $\mathrm{d}x$ we obtain:

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty} e^{-\pi x(t^2+n^2)} \mathrm{d}x = \sum_{n=1}^{\infty} b_n \int_0^{\infty} \frac{e^{-\pi\left(\frac{n^2}{x}+t^2 x\right)}}{\sqrt{x}} \mathrm{d}x + \int_0^{\infty} e^{-\pi t^2 x} \frac{\mathcal{P}(x)}{2} \mathrm{d}x$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} \frac{a_n}{\pi(t^2+n^2)} = \sum_{n=1}^{\infty} b_n \int_0^{\infty} \frac{e^{-\pi\left(\frac{n^2}{x}+t^2 x\right)}}{\sqrt{x}} \mathrm{d}x + \int_0^{\infty} e^{-\pi t^2 x} \frac{\mathcal{P}(x)}{2} \mathrm{d}x.$$

For the second integral change variable to $u = \sqrt{x}$ to obtain:

$$\int_0^{\infty} \frac{e^{-\pi\left(\frac{n^2}{x}+t^2 x\right)}}{\sqrt{x}} \mathrm{d}x = \int_0^{\infty} \frac{e^{-\pi\left(\frac{n^2}{u^2}+t^2 u^2\right)}}{u} 2u\mathrm{d}u = 2\int_0^{\infty} e^{-\pi\left(\frac{n^2}{u^2}+t^2 u^2\right)} \mathrm{d}u.$$

Change again variable, this time to $v = tu$:

$$2\int_0^{\infty} e^{-\pi\left(\frac{n^2}{u^2}+t^2 u^2\right)} \mathrm{d}u = 2\int_0^{\infty} e^{-\pi\left(\frac{t^2 n^2}{v^2}+v^2\right)} \frac{\mathrm{d}v}{t} = \frac{2}{t}\int_0^{\infty} e^{-\pi\left(\frac{t^2 n^2}{v^2}+v^2\right)} \mathrm{d}v = \frac{2}{t}\cdot\frac{1}{2} e^{-2\pi n t}.$$

Therefore:

$$\sum_{n=1}^{\infty} \frac{a_n}{\pi(t^2+n^2)} = \sum_{n=1}^{\infty} \frac{b_n}{t} e^{-2\pi n t} + \int_0^{\infty} e^{-\pi t^2 x} \frac{\mathcal{P}(x)}{2} \mathrm{d}x.$$

Notice that the last term is a sum of terms of the form

$$\int_0^{\infty} e^{-\pi t^2 x} x^a \log^b x \mathrm{d}x$$

where $b$ are integers and $\mathrm{Re}(a) > -1$, that is to say, it is a sum of terms of the form $t^{\alpha} \log^{\beta} t$, we will denote this sum $H(t)$.

Hence:

$$\sum_{n=1}^{\infty} \frac{a_n}{\pi(t^2+n^2)} = \sum_{n=1}^{\infty} \frac{b_n}{t} e^{-2\pi n t} + \frac{H(t)}{2}$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} \frac{a_n}{\pi(t^2+n^2)} - \frac{H(t)}{2} = \sum_{n=1}^{\infty} \frac{b_n}{t} e^{-2\pi n t}$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} \frac{a_n}{2\pi t}\left(\frac{1}{t+in} + \frac{1}{t-in}\right) - \frac{H(t)}{2} = \sum_{n=1}^{\infty} \frac{b_n}{t} e^{-2\pi n t}$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} a_n\left(\frac{1}{t+in} + \frac{1}{t-in}\right) - \pi t H(t) = 2\pi \sum_{n=1}^{\infty} b_n e^{-2\pi n t}.$$

In conclusion, the series on the left is a meromorphic function, with poles only at $\pm in$, but the function on the right is periodic with period $i$, hence so is

the function on the left (due to analytic continuation).
This implies that the residues at $ki$ and $(k+1)i$ are equal, but this are exactly $a_k$ and $a_{k+1}$, so $a_k = a_1$ for all $k$. We have therefore proven that:

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = a_1 \sum_{n=1}^{\infty} \frac{1}{n^s} = a_1 \zeta(s).$$

□

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**Thank you!**

**We hope this lesson has been beneficial in studying this interesting topic.**
**For more lessons or demonstrations, visit our website.**

## References


[1] Edward Charles Titchmarsh and David Rodney Heath-Brown. *The theory of the Riemann zeta-function*. Oxford university press, 1986.