In 1859, Bernhard Riemann published the article titled "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse," [4] which translates to "On the Number of Prime Numbers less than a Given Quantity" [3]. This work laid the foundation for centuries of future research.

In the following years, many mathematicians have studied this function. In today's lesson, we dive deeply into the proof of one of the most important results regarding the Riemann Zeta:

Theorem 1. *If* $Re(s) = 1$ *then* $\zeta(s) \neq 0$.

notice that this, using the reflection formula for the Riemann Zeta function, also implies that $\zeta(s) \neq 0$ if $Re(s) = 0$ and therefore that:

Theorem 2. *The nontrivial zeros of the Riemann zeta function are contained in the strip* $0 < Re(s) < 1$.

This theorem was proven independently by J.Hadamard and C.J. de La Vallée Poussin in 1896; this analysis concerns the former. This proof is also explored in [1] and [5].

Hadamard's proof appeared in the article "Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques" published in the "Bulletin de la Société mathématique de France" [2]; it can be divided in three claims:

Claim 1.

$$\mathcal{S} := \sum_p \frac{1}{p^\sigma} \tag{1}$$

grows indefinitely like $-\log(\sigma - 1)$ *for* $\sigma \in \mathbb{R}$, $\sigma > 1$, $\sigma \to 1$.

Claim 2. *For* $s = \sigma + it \in \mathbb{C}$ *and* $\sigma > 1$, $\sigma \to 1$*:*

$$Re(\log(\zeta(\sigma + it))) \approx \sum_p \frac{1}{p^\sigma} \cos(t \log p)$$

also if there exists $t_0 \in \mathbb{R}$ *such that* $\zeta(1 + it_0) = 0$ *then*

$$Re(\log(\zeta(\sigma + it_0))) \approx \sum_p \frac{1}{p^\sigma} \cos(t_0 \log p) \approx -\infty$$

Claim 3.

$$\mathcal{P} := \sum_p \frac{1}{p^\sigma} \cos(t_0 \log p) \tag{2}$$

does not diverge when $\sigma \in \mathbb{R}$, $\sigma > 1$, $\sigma \to 1$ *for any* $t_0 \in \mathbb{R}$.

Remark 1. *It is interesting to think what would mean for* $\sum_p \frac{1}{p^\sigma} \cos(t_0 \log p)$ *to diverge (say to* $-\infty$*); it would imply that* $\cos(t_0 \log p)$ *is nearly* -1 *for the overwhelming majority of the primes* p*. That would be an incredible regularity in the distribution of the numbers* $\log p$*, namely, that most of them lie near the points of the arithmetic progression* $(2n+1)t_0^{-1}\pi$*.*
While convenient, this is unfortunately false.

The bulk of the proof is demonstrating the second part of Claim 2 and then proving that supposing that the sum $\mathcal{P}$ diverges gives us a contradiction i.e. Claim 3, which consequentially implies that $\zeta(1 + it_0) \neq 0$ for any $t_0 \in \mathbb{R}$.

Remark 2. *Before we start, notice that the notation used by Hadamard in his original paper can be found a bit confusing, he first writes* s *for a real variable (while often in literature it refers to a complex one) and then uses the same* s *for the real part of a complex variable* $s + it$*; in an attempt to reduce confusion we opt to use the classical notation* $s = \sigma + it$*, where* s *is a complex variable,* $Re(s) = \sigma \in \mathbb{R}$ *and* $Im(s) = t \in \mathbb{R}$*.*

Most of the proof presented is a more detailed version of the ones offered in [1] and [5].

Proof. The article starts with an unusual definition of $\zeta(s)$:

Definition 1. *The function* $\zeta(s)$ *is defined, for* $Re(s) > 1$ *as:*

$$\log \zeta(s) = -\sum_p \log\left(1 - \frac{1}{p^s}\right) \tag{3}$$

where the sum is considered over all prime numbers p.
This is not the original definition but equivalent to it as it can simply be observed by using the Euler product of the Riemann Zeta function:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \tag{4}$$

which implies

$$\log(\zeta(s)) = \log\left(\prod_p \left(1 - \frac{1}{p^s}\right)^{-1}\right) = \sum_p \log\left(\left(1 - \frac{1}{p^s}\right)^{-1}\right) = -\sum_p \log\left(1 - \frac{1}{p^s}\right) \tag{5}$$

1 Claim 1

Hadamard proceeds by enunciating Claim 1, without giving a detailed proof. We instead take some time to explain this result clearly:

First notice that, using definition 3 and the Taylor expansion for $\log(1+x)$ we have:

$$\begin{aligned}
\log\zeta(\sigma) &= -\sum_p \log\left(1-\frac{1}{p^\sigma}\right) = -\sum_p \log\left(1+\left(-\frac{1}{p^\sigma}\right)\right) \\
&= -\sum_p\sum_{m=1}^{\infty}(-1)^{m+1}\frac{\left(-\frac{1}{p^\sigma}\right)^m}{m} = \sum_p\sum_{m=1}^{\infty}(-1)^{m+2}(-1)^m\frac{1}{mp^{m\sigma}} \\
&= \sum_p\sum_{m=1}^{\infty}\frac{1}{mp^{m\sigma}} = \sum_p\frac{1}{p^\sigma}+\sum_p\sum_{m=2}^{\infty}\frac{1}{mp^{m\sigma}} = \sum_p\frac{1}{p^\sigma}+f(\sigma),
\end{aligned} \tag{6}$$

with $f(\sigma)$ regular and bounded for $\sigma \geq 1$.

Remark 3. *It's important to note that the relation* $\log\zeta(s) = \sum_p\sum_{m=1}^{\infty}\frac{1}{mp^{ms}}$ *comes up very often when studying the Riemann zeta function.*

Remember that Riemann defined the ξ function as:

$$\xi(t) := \Gamma\left(\frac{s}{2}+1\right)(s-1)\pi^{-\frac{s}{2}}\zeta(s) \tag{7}$$

therefore:

$$\lim_{s\to 1}(s-1)\zeta(s) = \frac{\xi(1)\pi^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} = \frac{\frac{1}{2}\pi^{\frac{1}{2}}}{\frac{1}{2}\pi^{\frac{1}{2}}} = 1$$

i.e. $\zeta(s)$ has a simple pole at $s=1$.

Hence, as $\sigma \to 1$ for decreasing values:

$$\lim_{\sigma\to 1}\log(\zeta(\sigma)(\sigma-1)) = \lim_{\sigma\to 1}\left[\log\zeta(\sigma)+\log(\sigma-1)\right] = \log(1) = 0$$

$$\Downarrow$$

$$\lim_{\sigma\to 1}\left[\sum_p\frac{1}{p^\sigma}+f(\sigma)+\log(\sigma-1)\right] = 0$$

and therefore, for σ slightly larger than one:

$$\log\zeta(\sigma) \approx \mathcal{S} \approx -\log(\sigma-1) \tag{8}$$

which is exactly Claim 1.

2 Claim 2

The author proceeds to affirm that Claim 2 is true, this is once again not shown in the article, but we take the time to prove it properly:

First the fact that $Re(\log\zeta(\sigma+it))$ is approximately $\mathcal{P}$:

$$\begin{aligned}
&Re(\log\zeta(s)) = Re(\log\zeta(\sigma+it)) = Re\left(\sum_p \frac{1}{p^{\sigma+it}} + f(\sigma+it)\right)\\
&= Re\left(\sum_p \frac{1}{e^{(\sigma+it)\log p}}\right) + Re(f(\sigma+it))\\
&= Re\left(\sum_p \frac{e^{-it\log p}}{p^{\sigma}}\right) + Re(f(\sigma+it))\\
&= \sum_p \frac{1}{p^{\sigma}}(-\cos(t\log p)) + Re(f(\sigma+it))\\
&= \sum_p \frac{1}{p^{\sigma}}\cos(t\log p) + Re(f(\sigma+it)) = \mathcal{P} + Re(f(\sigma+it)).
\end{aligned} \tag{9}$$

$Re(f(\sigma+it))$ is once again bounded for $\sigma \geq 1$ and therefore the behavior of $Re(\log\zeta(s))$ is the same as $\mathcal{P}$ when $\sigma \to 1$.

Now, to prove the second claim, notice that 9 also implies that:

$$\mathcal{P} + Re(f(s)) - Re(\log(\zeta(s))) = 0$$

$$\Downarrow$$

$$\lim_{\sigma\to 1}\left[\sum_p \frac{1}{p^{\sigma}}\cos\left(t_0\log p\right) + Re(f(\sigma)) - Re(\log(\sigma-1))\right] = 0$$

but $Re(f(\sigma))$ is bounded and therefore:

$$\mathcal{P} = \sum_p \frac{1}{p^{\sigma}}\cos\left(t_0\log p\right) \approx \log(\sigma-1) \approx -\infty. \tag{10}$$

So we have proven that $\mathcal{P}$ grows indefinitely like $\log(\sigma-1)$ i.e. like $-\mathcal{S}$ when σ tends to 1, which combined with the first part of the claim, implies that:

$Re(\log\zeta(\sigma+it))$ can approach $-\infty$ as $\sigma\downarrow 1$ if and only if $\mathcal{P}\to-\infty$.

That is to say that if we were to find a zero of $\zeta(s)$ in the form $1+it$ then it would follow that:

$$\lim_{\sigma\downarrow 1}\mathcal{P} = -\infty. \tag{11}$$

3 Claim 3

Claim 3 is the one that is thoroughly proven by Hadamard, to understand his reasoning notice first that, by definition:

$$\mathcal{P} \leq \mathcal{S}$$

and therefore, 11 and 8 imply that, for any positive number ϵ we can find a σ, with $\sigma-1$ close enough to zero, such that:

$$\mathcal{P} < -(1-\epsilon)\mathcal{S} \tag{12}$$

this is never explicitly written in the paper, but it is a crucial part of the contradiction that Hadamard eventually finds.

Finding the contradiction is done as follows:
Consider α a small angle (nothing is specified about α, it is not wrong, but for the sake of clarity, we suppose $0 < \alpha < \frac{\pi}{4}$).
Distinguish two categories of prime numbers:
The prime numbers that satisfy

$$\frac{(2k+1)\pi - \alpha}{t_0} \leq \log p \leq \frac{(2k+1)\pi + \alpha}{t_0} \tag{13}$$

for some integer k and the prime numbers that don't for any integer k.
Call $\mathcal{S}_n$ and $\mathcal{P}_n$ the partial sums of the series $\mathcal{S}$ and $\mathcal{P}$ (i.e. the sums of the first n primes rather than the sum on all primes) and separate them as:

$$\mathcal{S}_n = \mathcal{S}_n' + \mathcal{S}_n''$$

$$\mathcal{P}_n = \mathcal{P}_n' + \mathcal{P}_n''$$

where $\mathcal{S}_n'$ and $\mathcal{P}_n'$ are only summed over the prime numbers belonging to the first category while $\mathcal{S}_n''$ and $\mathcal{P}_n''$ are only summed over the prime numbers belonging to the second category.
Denote with ρ_n the fraction:

$$\rho_n := \frac{\mathcal{S}_n'}{\mathcal{S}_n} \tag{14}$$

and notice that $0 < \rho_n < 1$, being a limited succession, we can extract a subsuccession that converges, consider that our succession ρ_n.

Hadamard claims that if there exists a complex number $1 + it_0$ such that $\zeta(1 + it_0) = 0$ then this succession has to tend to 1 as $\sigma \to 1$.

That is to say that, under the hypothesis that $\zeta(1+it_0) = 0$, for any number $\rho < 1$ and for any σ bigger than 1 but sufficiently close to 1, we should be able to find an n such that $\rho_{n_0} > \rho$ for all $n_0 > n$ (notice that in the original paper the author refers to this fixed complex number as $1+ti$ rather than $1+it_0$).

Let's understand why this is true:
Suppose that it isn't, meaning that $\zeta(1 + it_0) = 0$ for some complex number $1 + it_0$ but there is a number $\rho < 1$ such that

$$\rho_n \leq \rho \tag{15}$$

for all n bigger than a fixed n_0.

It is obvious that:

$$\frac{1}{p^s} \cos(t_0 \log p) \geq -\frac{1}{p^s}$$

and therefore, by definition:

$$\mathcal{P}_n' \geq -\mathcal{S}_n' = -\rho_n \mathcal{S}_n \tag{16}$$

while

$$-\cos(\alpha) > \cos(t_0 \log p)$$
$$\Updownarrow$$
$$\cos(\pi - \alpha) > \cos(t_0 \log p)$$

but $0 < \alpha < \frac{\pi}{4}$, therefore:

$$\cos(\pi - \alpha) > \cos(t_0 \log p) \iff 0 \leq (2k+1)\pi - \alpha < t_0 \log p \leq (2k+1)\pi + \alpha$$

for some integer k, that implies:

$$\frac{(2k+1)\pi - \alpha}{t_0} \leq \log p \leq \frac{(2k+1)\pi + \alpha}{t_0}$$

which would make p belong to the first category of primes.

We have proven that:

$$-\cos(\alpha) > \cos(t_0 \log p) \iff p \text{ belongs to the first category of primes} \tag{17}$$

and therefore:

$$\cos(t_0 \log p) \geq -\cos(\alpha) \iff p \text{ belongs to the second category.} \tag{18}$$

Hence

$$\mathcal{P}''_n \geq -\mathcal{S}''_n \cos(\alpha) = -(\mathcal{S}_n - \mathcal{S}'_n)\cos(\alpha) = -(1-\rho_n)\mathcal{S}_n \cos(\alpha). \tag{19}$$

now, notice that, if 15 is true then, given that $\cos(\alpha) < 1$:

$$\rho_n(1 - \cos(\alpha)) \leq \rho(1 - \cos(\alpha))$$
$$\Downarrow$$
$$\rho_n(1 - \cos(\alpha)) + \cos(\alpha) \leq \rho(1 - \cos(\alpha)) + \cos(\alpha)$$
$$\Downarrow$$
$$\rho_n + (1-\rho_n)\cos(\alpha) \leq \rho + (1-\rho)\cos(\alpha)$$
$$\Downarrow$$
$$-(\rho_n + (1-\rho_n)\cos(\alpha)) \geq -(\rho + (1-\rho)\cos(\alpha))$$

and therefore, using 16 and 19:

$$\begin{aligned} \mathcal{P}_n &= \mathcal{P}'_n + \mathcal{P}''_n \geq -\rho_n \mathcal{S}_n - (1-\rho_n)\mathcal{S}_n \cos(\alpha) \\ &= -(\rho_n + (1-\rho_n)\cos(\alpha))\mathcal{S}_n \geq -(\rho + (1-\rho)\cos(\alpha))\mathcal{S}_n = -\theta\mathcal{S}_n \end{aligned} \tag{20}$$

where $\theta = (\rho + (1-\rho)\cos(\alpha)) \leq 1$, this was supposed true for an infinite number of values of n, therefore we can consider the limit for $n \to \infty$ in 20 and write:

$$\mathcal{P} \geq -\theta\mathcal{S} \tag{21}$$

which is in contradiction with 12 and therefore in contradiction with the hypothesis that $\zeta(1+it_0)=0$.

We have therefore proven that the equality $\zeta(1+it_0)=0$ implies that the limit of ρ_n tends to 1 with s, consider this true then, and see how we obtain a contradiction:

Consider the series 2 for $2t_0$ rather than t_0 and call $\mathcal{Q}$ this new series.
Divide it as: $\mathcal{Q}_n = \mathcal{Q}_n' + \mathcal{Q}_n''$ in the same way we divided the series $\mathcal{P}_n$.
Similarly to how we obtained equations 16 and 19 we find that, if p belongs to the first category of primes then:

$$\frac{(2k+1)\pi-\alpha}{t_0} \le \log p \le \frac{(2k+1)\pi+\alpha}{t_0}$$

$$\Downarrow$$

$$(2k+1)\pi-\alpha \le t_0 \log p \le (2k+1)\pi+\alpha$$

$$\Downarrow$$

$$(2k+1)2\pi-2\alpha \le 2t_0 \log p \le (2k+1)2\pi+2\alpha$$

$$\Downarrow$$

$$\cos(2t_0 \log p) \ge \cos(2\alpha)$$

hence

$$\mathcal{Q}_n' \ge \mathcal{S}_n' \cos(2\alpha) = \rho_n \mathcal{S}_n \cos(2\alpha) \tag{22}$$

while, simply using that $\cos(2t_0 \log p) \ge -1$:

$$\mathcal{Q}_n'' \ge -\mathcal{S}_n'' = -(\mathcal{S}_n - \mathcal{S}_n') = -(1-\rho_n)\mathcal{S}_n. \tag{23}$$

and therefore

$$\mathcal{Q}_n = \mathcal{Q}_n' + \mathcal{Q}_n'' \ge \mathcal{S}_n \left[\rho_n \cos(2\alpha) - (1-\rho_n)\right] \tag{24}$$

so that we have:

$$\mathcal{Q}_n \ge \theta' \mathcal{S}_n, \tag{25}$$

where θ' denotes the number $\rho_n \cos(2\alpha) - (1-\rho_n)$ that is positive if we consider $\frac{1}{1+\cos(2\alpha)} < \rho_n \le 1$ (this is not a problem since $\rho_n \to 1$).

We have $\mathcal{S} \to \infty$ and therefore $\mathcal{Q} \to \infty$ for $\sigma \to 1$, but remember that $\mathcal{Q}$ is approximately $Re(\log \zeta(\sigma + i2t_0))$, so this would mean that $1 + i2t_0$ is a pole for the Riemann Zeta function, that we know is false. □

POSITIVE INCREMENT

Thank you!

We hope this lesson has been beneficial in studying this interesting topic.
For more lessons or demonstrations, visit our website.

References

[1] Harold M Edwards. *RiemannÆs zeta function*. Vol. 58. Academic press, 1974.

[2] Jacques Hadamard. "Sur la distribution des zéros de la fonction ζ (s) et ses conséquences arithmétiques". In: *Bulletin de la Societé mathematique de France* 24 (1896), pp. 199–220.

[3] Bernhard Riemann. "On the number of prime numbers less than a given quantity (ueber die anzahl der primzahlen unter einer gegebenen grösse)". In: *Monatsberichte der Berliner Akademie* (1859).

[4] Bernhard Riemann. "Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse". In: *Ges. Math. Werke und Wissenschaftlicher Nachlaß* 2.145-155 (1859), p. 2.

[5] Edward Charles Titchmarsh and David Rodney Heath-Brown. *The theory of the Riemann zeta-function*. Oxford university press, 1986.