This is an appendix to our Lesson: The Riemann Hypothesis and the Möbius function.

Here we give detailed proof of the following theorem:

Theorem 1. *Let $s \in \mathbb{C}$, $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$. Assuming true the Riemann Hypothesis, for every $\sigma > \frac{1}{2}$ and ϵ close to zero:*

$$\zeta(s) = \mathcal{O}(t^{\epsilon}) \tag{1}$$

and

$$\frac{1}{\zeta(s)} = \mathcal{O}(t^{\epsilon}). \tag{2}$$

This will be a simple corollary of:

Theorem 2. *Assuming true the Riemann Hypothesis, we have, for ϵ close to zero:*

$$\log \zeta(s) = \mathcal{O}\left((\log t)^{2-2\sigma+\epsilon}\right) \tag{3}$$

uniformly for $\frac{1}{2} < \sigma_0 \leq \sigma \leq 1$.

Proof. Start by applying the Borel-Carathéodory Theorem to the function $\log \zeta(s)$, it states:

Let f be an analytic function on a closed disc $\mathcal{D}$ of radius R and center c, call the whole circle $\mathcal{C}$ and suppose that $r < R$; then we have:

$$\max_{s \in \mathcal{D}} |f(s)| \leq \frac{2r}{R-r} \sup_{s \in \mathcal{C}} Re(f(s)) + \frac{R+r}{R-r}|f(c)|. \tag{4}$$

We will apply this Theorem to the two circles with center $c = 2 + it$ and radii $R = \frac{3}{2} - \frac{1}{2}\delta$ and $r = \frac{3}{2} - \delta$, where $0 < \delta < \frac{1}{2}$.

The larger circle is compact and there $\log \zeta(s)$ is analytic, therefore $Re(\log \zeta(s))$ is bounded, for reasons that will be clearer later we express this bound as:

$$Re(\log \zeta(s)) = \log |\zeta(s)| < A \log t$$

Hence, equation 4 with $R = \frac{3}{2} - \frac{1}{2}\delta$, $r = \frac{3}{2} - \delta$ and $c = 2 + it$ implies that on the smaller circle:

$$
\begin{aligned}
|\log\zeta(s)| &< \frac{3-2\delta}{\frac{1}{2}\delta} A\log t + \frac{3-\frac{3}{2}\delta}{\frac{1}{2}\delta}|\log\zeta(2+it)| \\
&= \frac{2}{\delta} A\log t\left(6 - 2\delta - \frac{3}{2}\delta\right) = \frac{2}{\delta}A\log t\left(6 - \frac{7}{2}\delta\right)
\end{aligned} \tag{5}
$$

now fixing a different A' this can be estimated again as:

$$
|\log\zeta(s)| < \frac{A'\log t}{\delta}.
$$

Use now Hadamard's **Three-circles Theorem**, it states:

Let $r_1 < r_2 < r_3$ be the radii of three concentric circles and $f(s)$ be an holomorphic function on the region:

$$
r_1 \leq |s| \leq r_3.
$$

If $M(r)$ denotes the maximum of $|f(s)|$ on the circle $|s| = r$, then:

$$
\log\left(\frac{r_3}{r_1}\right)\log M(r_2) \leq \log\left(\frac{r_3}{r_2}\right)\log M(r_1) + \log\left(\frac{r_2}{r_1}\right)\log M(r_3). \tag{6}
$$

We will use this theorem for the three circles C_1, C_2, C_3 with center $\sigma_1 + it$ $(1 < \sigma_1 \leq t)$, passing through the points $1 + \eta + it$, $\sigma + it$ and $\frac{1}{2} + \delta + it$ $(0 < \eta < \sigma_1 - 1;\ 0 < \sigma < \sigma_1)$ with these hypothesis the radii are:

$$
r_1 = \sigma_1 - 1 - \eta, \qquad r_2 = \sigma_1 - \sigma, \qquad r_3 = \sigma_1 - \frac{1}{2} - \delta.
$$

Therefore equation 6 applied to $f(s) = \log\zeta(s)$, calling the maxes of $|\log\zeta(s)|$ on the circles M_1, M_2, M_3, implies:

$$
\log\left(\frac{r_3}{r_1}\right)\log M_2 \leq \log\left(\frac{r_3}{r_2}\right)\log M_1 + \log\left(\frac{r_2}{r_1}\right)\log M_3
$$

$$
\Downarrow
$$

$$
\log M_2 \leq \frac{\log\left(\frac{r_3}{r_2}\right)}{\log\left(\frac{r_3}{r_1}\right)}\log M_1 + \frac{\log\left(\frac{r_2}{r_1}\right)}{\log\left(\frac{r_3}{r_1}\right)}\log M_3
$$

Notice now that:

$$
1 - \frac{\log\left(\frac{r_2}{r_1}\right)}{\log\left(\frac{r_3}{r_1}\right)} = \frac{\log\left(\frac{r_3}{r_1}\right) - \log\left(\frac{r_2}{r_1}\right)}{\log\left(\frac{r_3}{r_1}\right)} = \frac{\log r_3 - \log r_1 - \log r_2 + \log r_1}{\log\left(\frac{r_3}{r_1}\right)} = \frac{\log\left(\frac{r_3}{r_2}\right)}{\log\left(\frac{r_3}{r_1}\right)}.
$$

Therefore, calling

$$
a = \frac{\log\left(\frac{r_2}{r_1}\right)}{\log\left(\frac{r_3}{r_1}\right)}
$$

we have:

$$\log M_2 \le (1-a)\log M_1 + a \log M_3$$
$$\Downarrow$$
$$\log M_2 \le \log M_1^{1-a} + \log M_3^{a}$$
$$\Downarrow$$
$$M_2 \le M_1^{1-a} + M_3^{a}.$$

Focus on the term a:

$$a = \frac{\log\left(\frac{r_2}{r_1}\right)}{\log\left(\frac{r_3}{r_1}\right)} = \frac{\log\left(\frac{\sigma_1-\sigma}{\sigma_1-1-\eta}\right)}{\log\left(\frac{\sigma_1-\frac{1}{2}-\delta}{\sigma_1-1-\eta}\right)} = \frac{\log\left(\frac{\sigma_1-1-\eta+1+\eta-\sigma}{\sigma_1-1-\eta}\right)}{\log\left(\frac{\sigma_1-1-\eta+1+\eta-\frac{1}{2}-\delta}{\sigma_1-1-\eta}\right)}$$

$$= \frac{\log\left(1+\frac{1+\eta-\sigma}{\sigma_1-1-\eta}\right)}{\log\left(1+\frac{\eta+\frac{1}{2}-\delta}{\sigma_1-1-\eta}\right)}$$

use the fact that $\log(1+x) = x + \mathcal{O}(x)$ to obtain: (7)

$$= \frac{\frac{1+\eta-\sigma}{\sigma_1-1-\eta} + \mathcal{O}\left(\frac{1+\eta-\sigma}{\sigma_1-1-\eta}\right)}{\frac{\eta+\frac{1}{2}-\delta}{\sigma_1-1-\eta} + \mathcal{O}\left(\frac{\eta+\frac{1}{2}-\delta}{\sigma_1-1-\eta}\right)} = \frac{1+\eta-\sigma}{\eta+\frac{1}{2}-\delta} + \mathcal{O}\left(\frac{1}{\sigma_1}\right)$$

for $\delta \approx 0$ and $\eta \approx 0$ we are left with $2(1-\sigma)$, therefore

$$= 2 - 2\sigma + \mathcal{O}(\delta) + \mathcal{O}(\eta) + \mathcal{O}\left(\frac{1}{\sigma_1}\right).$$

Remembering equation 5, we know that $M_3 < A'\delta^{-1}\log t$ and in general:

$$\log\zeta(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n}\frac{1}{n^s} \tag{8}$$

where $\Lambda(n)$ is Von Mangoldt's function, defined as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime p and integer } k \ge 1 \\ 0 & \text{else} \end{cases} \tag{9}$$

Demonstration of equation 8 can be found on our site.

Notice that by its definition $\frac{\Lambda(n)}{\log n} \le 1$ and therefore:

$$M_1 = \max_{s\in C_1}\left|\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\log n}\frac{1}{n^s}\right| \le \sum_{n=2}^{\infty}\frac{1}{n^{1+\eta}} < K$$

for some constant K.

Hence:

$$|\log(\sigma+it)| < K^{1-a}\left(\frac{A'\log t}{\delta}\right)^{a} < C(\log t)^{2-2\sigma+\mathcal{O}(\delta)+\mathcal{O}(\eta)+\mathcal{O}\left(\frac{1}{\sigma_1}\right)}$$

and the result follows taking δ and η small enough and σ_1 big enough.

□

We can now prove the corollaries 1 and 2 by noting that the index $2-2\sigma+\epsilon$ of $\log t$ in 3 can be made smaller than one for a small enough ϵ and therefore, with a different $\epsilon' = 2-2\sigma+\epsilon < 1$:

$$-\epsilon' \log t < \log|\zeta(s)| < \epsilon' \log t \qquad (t > t_0(\epsilon))$$

which implies both 1 and 2.

POSITIVE INCREMENT

Thank you!

We hope this lesson has been beneficial in studying this interesting topic.
For more lessons or demonstrations, visit our website.