POSITIVE
INCREMENT

Definition 1. *The* ***Euler Totient Function*** *is defined for $n \geq 1$ as the number of positive integers not exceeding n which are relatively prime to n.*

That is to say:

$$\varphi(n) := \sum_{\substack{k=1 \\ (k,n)=1}}^{n} 1. \tag{1}$$

Remark 1. *Most of the following results are a more detailed explanation of what is contained in [1].*

Theorem 1. *For $n \geq 1$ we have:*

$$\sum_{d|n} \varphi(d) = n. \tag{2}$$

Proof. Let S denote the set of numbers from 1 to n: $S = \{1, 2, \cdots, n\}$. For each divisor d of n define:

$$A(d) := \{k : (k,n) = d, 1 \leq k \leq n\}. \tag{3}$$

That is the set of those elements of S which have greatest common denominator with n equal to d. These sets are disjoint and their union is the whole S. Therefore:

$$\sum_{d|n} |A(d)| = n. \tag{4}$$

Remember the following properties of the greatest common denominator:

$$(k,n) = d \iff \left(\frac{k}{d}, \frac{n}{d}\right) = 1, \quad \text{and} \quad 0 < k \leq n \iff 0 < \frac{k}{d} \leq \frac{n}{d}.$$

Therefore, if $q = \frac{k}{d}$, the number of integers satisfying $0 < q \leq \frac{n}{d}$ and $\left(q, \frac{n}{d}\right) = 1$ is exactly the number of elements of $A(d)$. By definition, this is also $\varphi\left(\frac{n}{d}\right)$. Hence, using equation 4:

$$|A(d)| = \varphi\left(\frac{n}{d}\right) \Rightarrow \sum_{d|n} \varphi\left(\frac{n}{d}\right) = n.$$

But this is equivalent to $\sum_{d|n} \varphi(d) = n$ because when d runs through all divisors of n, so does $\frac{n}{d}$. □

Theorem 2. *For $n \geq 1$ we have:*

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) \tag{5}$$

Proof. Let's start by proving the formula for $n = p^k$ for some p prime and $k \geq 1$ integer.

The numbers not coprime to n smaller than n are the multiples of p: $p, 2p, \cdots, p^{k-1}p = p^k$. There are p^{k-1} such multiples, therefore:

$$\varphi\left(p^k\right) = p^k - p^{k-1} = p^k \left(1 - \frac{1}{p}\right) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

To prove the general case it suffices to demonstrate that:

$$\varphi(mn) = \varphi(m)\varphi(n) \qquad \text{if } (m,n) = 1. \tag{6}$$

This is a corollary of the Chinese Remainder Theorem, as it states that there is a bijection between $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ and $\mathbb{Z}/nm\mathbb{Z}$, but of course, $|\mathbb{Z}/n\mathbb{Z}| = \varphi(n)$,$|\mathbb{Z}/m\mathbb{Z}| = \varphi(m)$ and $|\mathbb{Z}/nm\mathbb{Z}| = \varphi(nm)$.

To conclude, for any integer n with prime decomposition $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, we have:

$$\begin{aligned} \varphi(n) &= \varphi\left(p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}\right) = \varphi\left(p_1^{a_1}\right)\varphi\left(p_2^{a_2}\right) \cdots \varphi\left(p_k^{a_k}\right) \\ &= p_1^{a_1}\left(1 - \frac{1}{p_1}\right) p_2^{a_2}\left(1 - \frac{1}{p_2}\right) \cdots p_k^{a_k}\left(1 - \frac{1}{p_k}\right) = n \prod_{p|n}\left(1 - \frac{1}{p}\right). \end{aligned} \tag{7}$$

□

References

[1] Tom M Apostol. *Introduction to analytic number theory.* Springer Science & Business Media, 2013.