Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1. *The Riemann* ξ *function, defined as:*

$$\xi(s) := \frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)$$

satisfies the equation

$$\xi(s) = \xi(1-s) \tag{2}$$

for $s \neq 0, 1$.

This is a classical result that can be found in many textbooks, the demonstration that follows is a detailed explanation of the proof that appears in [1].

Proof. The demonstration relies heavily on the equation:

$$\zeta(s) = 2(2\pi)^{s-1}\sin\left(\frac{\pi}{2}s\right)\Gamma(1-s)\zeta(1-s) \tag{3}$$

this is not an obvious result, the full proof can be found on our site.

Start by computing $\xi(1-s)$, using its definition:

$$\xi(1-s) = \frac{1}{2}(1-s)(1-s-1)\Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}\zeta(1-s)$$

$$\Downarrow$$

$$\xi(1-s) = \frac{1}{2}s(s-1)\Gamma\left(\frac{1-s}{2}\right)\pi^{\frac{s-1}{2}}\zeta(1-s).$$

The definition of ξ implies:

$$\zeta(s) = \frac{2\xi(s)\pi^{\frac{s}{2}}}{s(s-1)\Gamma\left(\frac{s}{2}\right)}$$

therefore using 3:

$$\begin{gathered}\frac{2\xi(s)\pi^{\frac{s}{2}}}{s(s-1)\Gamma\left(\frac{s}{2}\right)} = 2(2\pi)^{s-1}\sin\left(\frac{\pi}{2}s\right)\Gamma(1-s)\zeta(1-s)\\ \Downarrow \\ \xi(s) = s(s-1)2^{s-1}\pi^{\frac{s}{2}-1}\sin\left(\frac{\pi}{2}s\right)\Gamma(1-s)\Gamma\left(\frac{s}{2}\right)\zeta(1-s).\end{gathered} \tag{4}$$

Remember now Legendre's duplication formula for the Gamma function:

$$2\pi^{\frac{1}{2}}2^{-2s}\Gamma(2s) = \Gamma(s)\Gamma\left(s+\frac{1}{2}\right)$$

when s is replaced by $\frac{1-s}{2}$ this becomes

$$\begin{gathered}2\pi^{\frac{1}{2}}2^{s-1}\Gamma(1-s) = \Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1-s+1}{2}\right)\\ \Downarrow \\ 2^{s}\pi^{\frac{1}{2}}\Gamma(1-s) = \Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\\ \Downarrow \\ \Gamma(1-s) = 2^{-s}\pi^{-\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1-\frac{1}{2}s\right).\end{gathered} \tag{5}$$

Combine this with the known property of the Gamma function:

$$\begin{gathered}\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}\\ \Downarrow \\ \sin\left(\pi\frac{s}{2}\right) = \frac{\pi}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}\end{gathered}$$

to find:

$$\begin{gathered}\Gamma(1-s) = 2^{-s}\pi^{-\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\\ \Downarrow \\ \Gamma(1-s)\sin\left(\pi\frac{s}{2}\right) = 2^{-s}\pi^{-\frac{1}{2}}\pi\frac{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}\\ \Downarrow \\ \Gamma(1-s)\sin\left(\pi\frac{s}{2}\right) = 2^{-s}\pi^{\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.\end{gathered} \tag{6}$$

Remembering equation 4 we have:

$$
\begin{aligned}
\xi(s) &= s(s-1)2^{s-1}\pi^{\frac{s}{2}-1}\sin\left(\frac{\pi}{2}s\right)\Gamma(1-s)\Gamma\left(\frac{s}{2}\right)\zeta(1-s) \\
&= s(s-1)2^{s-1}\pi^{\frac{s}{2}-1}2^{-s}\pi^{\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\Gamma\left(\frac{s}{2}\right)\zeta(1-s) \\
&= \frac{1}{2}s(s-1)\Gamma\left(\frac{1-s}{2}\right)\pi^{\frac{s-1}{2}}\zeta(1-s)
\end{aligned} \tag{7}
$$

which is exactly 2. □

References

[1] Tom M Apostol. *Introduction to analytic number theory*. Springer Science & Business Media, 2013.