Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$*.*

Theorem 1.

$$\zeta(s) = \frac{1}{(s-1)} + \frac{\sin(\pi s)}{\pi(s-1)} \int_0^\infty x^{1-s} \left(\frac{1}{1+x} - \psi'(1+x) \right) dx$$

for $0 < Re(s) < 2$, $s \neq 1$, *where* $\psi(x) := \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ *is the Digamma function.*

The following is a more detailed version of the proof that appears in De Bruijn's original paper [1]. We will use the common notation $Re(s) = \sigma$.

Proof. Start by considering the equation

$$\zeta(1+s) = \frac{\sin(\pi s)}{\pi s} \int_0^\infty \psi'(1+x) \frac{dx}{x^s},$$

true for $0 < Re(s) < 1$, this is not an obvious relation, the proof can be found on our site.

Evaluating the formula for s yields:

$$\zeta(s) = \frac{\sin(\pi(s-1))}{\pi(s-1)} \int_0^\infty \psi'(1+x) \frac{dx}{x^{s-1}},$$

considering now that, for $1 < \sigma < 2$:

$$\frac{1}{s-1} = \frac{\sin(\pi s)}{\pi(1-s)} \int_0^\infty \frac{dx}{x^{s-1}(1+x)}$$

we have:

$$\zeta(s) - \frac{1}{s-1} = \frac{\sin(\pi(s-1))}{\pi(s-1)} \int_0^\infty \psi'(1+x)\frac{dx}{x^{s-1}} - \frac{\sin(\pi s)}{\pi(1-s)} \int_0^\infty \frac{dx}{x^{s-1}(1+x)}$$
$$= \frac{-\sin(\pi s)}{\pi(s-1)} \int_0^\infty x^{1-s} \cdot \psi'(1+x)dx + \frac{\sin(\pi s)}{\pi(s-1)} \int_0^\infty \frac{x^{1-s}}{(1+x)}dx$$
$$= \frac{\sin(\pi s)}{\pi(s-1)} \left[\int_0^\infty x^{1-s} \frac{dx}{(1+x)} - \int_0^\infty x^{1-s} \cdot \psi'(1+x)dx \right]$$
$$= \frac{\sin(\pi s)}{\pi(s-1)} \int_0^\infty x^{1-s} \left(\frac{1}{1+x} - \psi'(1+x) \right) dx. \tag{2}$$

Therefore:

$$\zeta(s) = \frac{1}{(s-1)} + \frac{\sin(\pi s)}{\pi(s-1)} \int_0^\infty x^{1-s} \left(\frac{1}{1+x} - \psi'(1+x) \right) dx$$

Now notice that the integral $\int_0^\infty \frac{x^{1-s}}{1+x}dx$ is uniformly convergent in every region lying between the lines $\sigma = \delta$ and $\sigma = 2 - \delta$ and the same is true for $\int_0^\infty x^{1-s}\psi'(1+x)dx$ given that $\psi'(x) = \mathcal{O}\left(\frac{1}{x^2}\right)$.
Hence the formula is actually valid for $0 < \sigma < 2$, $s \neq 1$. □

References

[1] NG de Bruijn. "Integralen voor de zeta -functie van Riemann". In: *Mathematica: tijdschrift voor studeerenden voor de acten wiskunde MO en voor studeerenden aan Universiteiten. Afdeeling B* 5 (1937), pp. 170–180.