Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1.

$$\zeta(1+s) = \frac{\sin(\pi s)}{\pi s} \int_0^\infty \psi'(1+x) \frac{dx}{x^s}$$

for $0 < Re(s) < 1$, *here* $\psi(x) := \frac{d}{dx}[\ln \Gamma(x)] = \frac{\Gamma'(x)}{\Gamma(x)}$ *is the Digamma function.*

This formula was first proven by De Brujin in [1]. What follows is a translation of the article to which we added a few details necessary for an easier understanding. We will use the common notation $Re(s) = \sigma$.

Proof. Let's start by proving that, for any $k \neq 0$:

$$\frac{1}{k^{s+1}} = \frac{\sin(\pi s)}{\pi s} \int_0^\infty \frac{dx}{x^s (k+x)^2} \tag{2}$$

First, changing variable to $x = kt$:

$$\int_0^\infty \frac{dx}{x^s(x+k)} = \int_0^\infty \frac{kdt}{(kt)^s(kt+k)} = \frac{1}{k^s} \int_0^\infty \frac{dt}{t^s(t+1)}.$$

The integral $\int_0^\infty \frac{dt}{t^s(t+1)}$ is a special case of the so called "Beta function" defined as

$$B(z_1, z_2) := \int_0^\infty \frac{t^{z_1 - 1}}{(1+t)^{z_1+z_2}} dt,$$

it is a known fact that

$$B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)}.$$

Remark 1. *At the moment of writing Positive Increment does not contain a lesson regarding the Beta Function, we apologize, and we are trying to compensate as soon as possible.*

In our case:

$$\int_0^\infty \frac{dx}{x^s(x+k)} = \frac{1}{k^s}\int_0^\infty \frac{dt}{t^s(t+1)} = \frac{1}{k^s}B(1-s,s) = \frac{1}{k^s}\frac{\Gamma(1-s)\Gamma(s)}{\Gamma(1)} = \frac{1}{k^s}\frac{\pi}{\sin(\pi s)}$$

in the last equality we used the known property of the Gamma function: $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$.

We have therefore proven that:

$$\frac{1}{k^s} = \frac{\sin(\pi s)}{\pi}\int_0^\infty \frac{dx}{x^s(x+k)}. \tag{3}$$

Now simply differentiate both sides with respect to k to obtain 2:

$$\frac{1}{k^s} = \frac{\sin(\pi s)}{\pi}\int_0^\infty \frac{dx}{x^s(x+k)}.$$
$$\Downarrow$$
$$\frac{-s}{k^{s+1}} = \frac{\sin(\pi s)}{\pi}\int_0^\infty \frac{-dx}{x^s(x+k)^2}$$
$$\Downarrow$$
$$\frac{1}{k^{s+1}} = \frac{\sin(\pi s)}{\pi s}\int_0^\infty \frac{dx}{x^s(x+k)^2}$$

here we used Leibniz's Integral rule to differentiate under the integral sign.

Equation 2 is true for any $k \neq 0$, consider those for k an integer from $k = 1$ to $k = n$ and sum them to obtain:

$$\sum_{k=1}^{n}\frac{1}{k^{s+1}} = \frac{\sin(\pi s)}{\pi s}\int_0^\infty \sum_{k=1}^{n}\frac{1}{(k+x)^2}\frac{dx}{x^s}. \tag{4}$$

The derivative of the Digamma function is:

$$\psi'(x) = \sum_{k=0}^{\infty}\frac{1}{(x+k)^2}$$

therefore:

$$\psi'(1+x) = \sum_{k=0}^{\infty}\frac{1}{(k+1+x)^2} = \sum_{k=1}^{\infty}\frac{1}{(k+x)^2}. \tag{5}$$

Remember now that, by definition, $\zeta(1+s) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^{s+1}}$, so that equation 4 implies, for $0 < \sigma < 1$:

$$\begin{aligned}\zeta(1+s) &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^{s+1}} = \lim_{n\to\infty}\frac{\sin(\pi s)}{\pi s}\int_0^\infty \sum_{k=1}^{n}\frac{1}{(k+x)^2}\frac{dx}{x^s} \\ &= \frac{\sin(\pi s)}{\pi s}\int_0^\infty \sum_{k=1}^{\infty}\frac{1}{(k+x)^2}\frac{dx}{x^s} = \frac{\sin(\pi s)}{\pi s}\int_0^\infty \psi'(1+x)\frac{dx}{x^s}.\end{aligned} \tag{6}$$

here the exchange of limit and integral is justified using Lebesgue's dominated convergence theorem with the fact that: $|\sum_{k=1}^{n}\frac{1}{k+x}| \leq |\psi'(1+x)|$. □

Corollary 1.

$$\zeta(1+s) = \frac{\sin(\pi s)}{\pi} \int_0^\infty (\gamma + \psi(1+x)) \frac{dx}{x^{s+1}}$$

for $0 < Re(s) < 1$ *and* $\gamma := \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n)\right)$ *is Euler's constant.*

Proof. This formula is obtained using integration by parts, being careful to choose a specific primitive of $\psi'(1+x)$:

$$\psi'(1+x) = \frac{d}{dx}(\psi(1+x) - \psi(1)) = \frac{d}{dx}(\psi(1+x) + \gamma)$$

as it is a known property of the Digamma function that $\psi(1) = -\gamma$.
This choice is necessary to make sure that the term outside the integral is null.

$$\begin{aligned} \zeta(1+s) &= \frac{\sin(\pi s)}{\pi s} \int_0^\infty \left[\frac{d}{dx}(\psi(1+x) + \gamma)\right] \frac{dx}{x^s} \\ &= \frac{\sin(\pi s)}{\pi s} \left| \frac{\psi(1+x) + \gamma}{x^s} \right|_0^\infty - \frac{\sin(\pi s)}{\pi s}(-s) \int_0^\infty (\psi(1+x) + \gamma) \frac{dx}{x^{s+1}} \\ &= \frac{\sin(\pi s)}{\pi} \int_0^\infty (\psi(1+x) + \gamma) \frac{dx}{x^{s+1}} \end{aligned} \tag{7}$$

the term outside the integral is zero, due to the fact that $\psi(1) = -\gamma$ and $0 < \sigma < 1$.

□

References

[1] NG de Bruijn. "Integralen voor de zeta -functie van Riemann". In: *Mathematica: tijdschrift voor studeerenden voor de acten wiskunde MO en voor studeerenden aan Universiteiten. Afdeeling B* 5 (1937), pp. 170–180.