Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1.

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s} + \sum_{j=1}^{k} \binom{s+2j-2}{2j-1} \frac{B_{2j}}{2j} N^{1-s-2j} - \binom{s+k}{k+1} \int_N^{\infty} \frac{\overline{B}_{k+1}(x)}{x^{s+k+1}} dx \tag{2}$$

where $Re(s) > -2k$*,* $N, k = 1, 2, \cdots$*,* $B_n(x)$ *are the Bernoulli polynomials,* $\overline{B}_n(x) := B_n(x - [x])$ *are the Bernoulli periodic functions while the function* $[x] := \max\{k \in \mathbb{Z} : k \leq x\}$*, is known as "floor" of* x *or integer part of* x*.*

Proof. Start by considering the equation

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s \int_N^{\infty} \frac{x - [x]}{x^{s+1}} dx$$

this is known as Representation by Euler-Maclaurin-formula.

By adding and subtracting $\frac{1}{2}$ to $x - [x]$ we obtain the first Bernoulli periodic function $\overline{B}_1(x) = x - [x] - \frac{1}{2}$:

$$\begin{aligned} \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s \int_N^{\infty} \frac{x - [x]}{x^{s+1}} dx &= \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s \int_N^{\infty} \frac{\overline{B}_1(x) + \frac{1}{2}}{x^{s+1}} dx \\ &= \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s} - s \int_N^{\infty} \frac{\overline{B}_1(x)}{x^{s+1}} dx \end{aligned} \tag{3}$$

where we used that $-\frac{s}{2}\int_N^\infty \frac{1}{x^{s+1}} = -\frac{1}{2}N^{-s}$.

To obtain the result we will use the following properties of the Bernoulli Polynomials:

$$B_n(0) = B_n \tag{4}$$

$$\overline{B}_{n-1}(x) = \frac{d}{dx}\left(\frac{B_n(x)}{n}\right) \tag{5}$$

Where B_n is the $n-th$ Bernoulli number, proofs can be found in [1].

We proceed now using integration by parts, remembering that by equation 5 we have $\int \overline{B}_1(x)dx = \frac{\overline{B}_2(x)}{2}$:

$$\int_N^\infty \frac{\overline{B}_1(x)}{x^{s+1}}dx = \left[\frac{1}{2}\frac{\overline{B}_2(x)}{x^{s+1}}\right]_N^\infty - \frac{s+1}{2}\int_N^\infty \frac{\overline{B}_2(x)}{x^{s+2}}dx$$

if we suppose $Re(s) > -2$ than $\lim_{x\to\infty} \frac{\overline{B}_1(x)}{x^{s+1}} = 0$ given that the k-th Bernoulli Polynomial has degree k, also by definition of $\overline{B}_k(x)$:

$$\overline{B}_k(N) = B_k(0) = B_k$$

for any integer N.

Therefore

$$\left[\frac{1}{2}\frac{\overline{B}_2(x)}{x^{s+1}}\right]_N^\infty = -\frac{B_2}{2}N^{-s-1}$$

and we obtain:

$$\begin{aligned}
&\sum_{n=1}^N \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s} - s\int_N^\infty \frac{\overline{B}_1(x)}{x^{s+1}}dx \\
&= \sum_{n=1}^N \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s} + s\frac{B_2}{2}N^{-s-1} - \frac{s(s+1)}{2}\int_N^\infty \frac{\overline{B}_2(x)}{x^{s+2}}dx \\
&= \sum_{n=1}^N \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s} + \binom{s}{1}\frac{B_2}{2}N^{-s-1} - \binom{s+1}{2}\int_N^\infty \frac{\overline{B}_2(x)}{x^{s+2}}dx
\end{aligned} \tag{6}$$

so exactly our formula for $k = 1$.

The general formula is obtained by integrating by parts k times.
Notice first that $B_{2k+1} = 0$, therefore the first term of the integration by parts is zero when we iterate for an even number of times, this explains why the first sum of the general formula only contains even indexes.

Summing up, under the hypothesis that $Re(s) > -2k$, the term outside the integral after the $k-$th iteration will be:

$$\begin{cases} 0 & \text{after an even number } k \text{ of iterations} \\ \binom{s+2k-2}{2k-1}\frac{B_{2k}}{2k}N^{1-s-2k} & \text{after an odd number } k \text{ of iterations} \end{cases} \tag{7}$$

while the integral part will always be multiplied by factor $\frac{s+k}{k}$ that times the previous $\binom{s+k-1}{k-1}$ gives us exactly $\binom{s+k}{k+1}$. □

Corollary 1.

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{j=1}^{k} \binom{s+2j-2}{2j-1} \frac{B_{2j}}{2j} - \binom{s+k}{k+1} \int_1^\infty \frac{\overline{B}_{k+1}(x)}{x^{s+k+1}} dx \qquad (8)$$

where $Re(s) > -2k,\ k = 1, 2, \cdots$.

Proof. To obtain this we only need to consider $N = 1$ in Theorem 1:

$$\begin{aligned}
\zeta(s) &= \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s} + \sum_{j=1}^{k} \binom{s+2j-2}{2j-1} \frac{B_{2j}}{2j} N^{1-s-2j} - \binom{s+k}{k+1} \int_N^\infty \frac{\overline{B}_{k+1}(x)}{x^{s+k+1}} dx \\
&= 1 + \frac{1}{s-1} - \frac{1}{2} + \sum_{j=1}^{k} \binom{s+2j-2}{2j-1} \frac{B_{2j}}{2j} - \binom{s+k}{k+1} \int_1^\infty \frac{\overline{B}_{k+1}(x)}{x^{s+k+1}} dx \\
&= \frac{1}{s-1} + \frac{1}{2} + \sum_{j=1}^{k} \binom{s+2j-2}{2j-1} \frac{B_{2j}}{2j} - \binom{s+k}{k+1} \int_1^\infty \frac{\overline{B}_{k+1}(x)}{x^{s+k+1}} dx.
\end{aligned} \qquad (9)$$

□

References

[1] Omran Kouba. "Lecture notes, Bernoulli polynomials and applications". In: *arXiv preprint arXiv:1309.7560* (2013).