Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1.

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s\int_N^{\infty} \frac{x-[x]}{x^{s+1}}dx \tag{2}$$

for $Re(s) > 0$, $N = 1, 2, \cdots$ *and* $[x] := \max\{k \in \mathbb{Z} : k \leq x\}$, *this is known as "floor" of* x *or integer part of* x.

Proof. We take inspiration from [1], in this book the author proves that

$$\zeta(s,a) = \sum_{n=0}^{N} \frac{1}{(n+a)^s} + \frac{(N+a)^{1-s}}{s-1} - s\int_N^{\infty} \frac{x-[x]}{(x+a)^{s+1}}dx \tag{3}$$

where $\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$ is called the Hurwitz zeta function, it is obvious that $\zeta(s) = \zeta(s,1)$ so that equation 3 implies equation 2, we will rewrite the proof for the case $a = 1$.

Start by applying Euler's summation formula:

$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)dt + \int_y^x (t-[t])f'(t)dt + f(x)(x-[x]) + f(y)(y-[y])$$

true for any function f with continuous derivative f' on the interval $[y,x]$ and $0 < y < x$, proof of this can also be found in Apostol's book [1], Theorem 3.1.

Apply this to the function $f(t) = (t+1)^{-s}$ and $x, y \in \mathbb{Z}_{>0}$ so that the function is continuous and with continuous derivative in $[y,x]$ and $x - [x] = y - [y] = 0$ by definition of $[x]$.

We obtain

$$\sum_{y<n\le x} \frac{1}{(n+1)^s} = \int_y^x \frac{dt}{(t+1)^s} - s\int_y^x \frac{t-[t]dt}{(t+1)^{s+1}}$$

set $y = N$ and $x \to \infty$ with $Re(s) > 1$

$$\sum_{N+1}^{\infty} \frac{1}{(n+1)^s} = \int_N^{\infty} \frac{1}{(t+1)^s} - s\int_N^{\infty} \frac{t-[t]}{(t+1)^{s+1}}dt.$$

Now, by definition of the zeta function: $\zeta(s) = \sum_{n=0}^{N} \frac{1}{(n+1)^s} + \sum_{N+1}^{\infty} \frac{1}{(n+1)^s}$ so:

$$\zeta(s) - \sum_{n=0}^{N} \frac{1}{(n+1)^s} = \sum_{N+1}^{\infty} \frac{1}{(n+1)^s} = \int_N^{\infty} \frac{1}{(t+1)^s} - s\int_N^{\infty} \frac{t-[t]}{(t+1)^{s+1}}dt$$

the first integral is easily solvable:

$$\zeta(s) - \sum_{n=0}^{N} \frac{1}{(n+1)^s} = \frac{(N+1)^{1-s}}{s-1} - s\int_N^{\infty} \frac{t-[t]}{(t+1)^{s+1}}dt$$

$$\Downarrow$$

$$\zeta(s) = \sum_{n=0}^{N} \frac{1}{(n+1)^s} + \frac{(N+1)^{1-s}}{s-1} - s\int_N^{\infty} \frac{t-[t]}{(t+1)^{s+1}}dt$$

$$\Downarrow$$

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s\int_{N-1}^{\infty} \frac{t-[t]}{(t+1)^{s+1}}dt.$$

Now simply call $x = t + 1$ to obtain:

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s\int_N^{\infty} \frac{x-1-[x]+1}{x^{s+1}}dx = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s\int_N^{\infty} \frac{x-[x]}{x^{s+1}}dx.$$

Which proves the theorem for $Re(s) > 1$, $N = 1, 2, \cdots$.

If $Re(s) \ge \delta > 0$ the integral is dominated by $\int_N^{\infty} \frac{1}{(t+a)^{\delta+1}}dt$ so that we have uniform convergence for $Re(s) > 0$. □

Corollary 1.

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s\int_1^{\infty} \frac{x-[x]-\frac{1}{2}}{x^{s+1}}dx = \frac{1}{s-1} + \frac{1}{2} - s\int_1^{\infty} \frac{\overline{B}_1(x)}{x^{s+1}}dx \quad (4)$$

where $\overline{B}_1(x) = x - [x] - \frac{1}{2}$ *is the first "Bernoulli periodic function"* ($\overline{B}_n(x) := B_n(x - [x])$).

Proof. Simply fix $N = 1$ in Theorem 1 to obtain:

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^\infty \frac{x-[x]}{x^{s+1}}dx$$

now add and subtract $\frac{1}{2}$ to $x - [x]$

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^\infty \frac{x-[x]+\frac{1}{2}-\frac{1}{2}}{x^{s+1}}dx$$

$$\Downarrow$$

$$\zeta(s) = 1 + \frac{1}{s-1} - \frac{s}{2}\int_1^\infty \frac{1}{x^{s+1}}dx - s\int_1^\infty \frac{x-[x]-\frac{1}{2}}{x^{s+1}}dx$$

$$\Downarrow$$

$$\zeta(s) = 1 + \frac{1}{s-1} - \frac{1}{2} - s\int_1^\infty \frac{x-[x]-\frac{1}{2}}{x^{s+1}}dx$$

$$\Downarrow$$

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s\int_1^\infty \frac{x-[x]-\frac{1}{2}}{x^{s+1}}dx$$

□

References

[1] Tom M Apostol. *Introduction to analytic number theory.* Springer Science & Business Media, 2013.