POSITIVE INCREMENT

Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1. *The Riemann Zeta function satisfies the equation:*

$$\log \zeta(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log(n)} \frac{1}{n^s} \tag{2}$$

for $Re(s) > 1$. *Here* $\Lambda(n)$ *is the Von Mangoldt Function, defined for* $n \in \mathbb{N}$ *as:*

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Proof. At the core of this proof there's the Euler product for the Riemann Zeta Function:

$$\zeta(s) = \prod_p \left(\frac{1}{1 - p^{-s}} \right) \tag{4}$$

the product runs over all prime numbers p and $Re(s) > 1$, proof of this formula can be found on our site.

Equation 4 implies that:

$$\log \zeta(s) = -\sum_p \log(1 - p^{-s}) = -\sum_p \log(1 + (-p^{-s})) \tag{5}$$

where again the product is over all prime numbers p and $Re(s) > 1$.

Given that $|p^{-s}| < 1$ for $Re(s) > 1$, we are allowed to use the Taylor expansion for the logarithmic function:

$$\begin{aligned} \log \zeta(s) &= -\sum_{p} \log(1 + (-(p^{-s}))) = -\sum_{p} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(-(p^{-s}))^k}{k} \\ &= \sum_{p} \sum_{k=1}^{\infty} (-1)^k (-1)^k \frac{p^{-ks}}{k} = \sum_{p} \sum_{k=1}^{\infty} \frac{p^{-ks}}{k}. \end{aligned} \tag{6}$$

Call now $n = p^k$ for any p prime number and k integer with $k \geq 1$, then $k = \log_p(n)$ and we obtain:

$$\log \zeta(s) = \sum_{n=p^k} \frac{n^{-s}}{\log_p(n)} = \sum_{n=p^k} \frac{n^{-s}}{\frac{\log(n)}{\log(p)}} = \sum_{n=p^k} \frac{n^{-s} \log(p)}{\log(n)} \tag{7}$$

where we used the known property of logarithms, $\log_a(b) = \frac{\log_c(b)}{\log_c(a)}$ and therefore $\log_p(n) = \frac{\log(p)}{\log(n)}$.

Now it is simply a matter of noticing that, by definition:

$$\sum_{n=p^k} \frac{n^{-s} \log(p)}{\log(n)} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log(n)} \frac{1}{n^s}$$

$$\Downarrow$$

$$\log \zeta(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log(n)} \frac{1}{n^s}.$$

□