Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1.

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + \sum_{m=1}^{n} \frac{B_{2m}}{(2m)!} \frac{\Gamma(s+2m-1)}{\Gamma(s)} + \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} - \sum_{m=1}^{n} \frac{B_{2m}}{(2m)!} x^{2m-1} \right) \frac{x^{s-1}}{e^x} dx$$

for $Re(s) > -2n$ and $n = 1, 2, 3, \cdots$; here B_n is the n-th Bernoulli number.

Remark 1. *The fraction $\frac{\Gamma(s+n)}{\Gamma(s)}$ is sometimes abbreviated using the Pochhamer's symbol or shifted factorial $(s)_n$.*

Proof. Start by considering the formula

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right) \frac{x^{s-1}}{e^x} dx$$

true for $Re(s) > 1$, proof of this equation can be found on our site.

Add and subtract the sum $\sum_{m=1}^{n} \frac{B_{2m}}{(2m)!} x^{2m-1}$ in the integral to obtain:

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} + \sum_{m=1}^{n} \frac{B_{2m}}{(2m)!} x^{2m-1} - \sum_{m=1}^{n} \frac{B_{2m}}{(2m)!} x^{2m-1} \right) \frac{x^{s-1}}{e^x} dx$$

remembering the definition of the Gamma function:

$$\Gamma(s) = \int_0^\infty \frac{x^{s-1}}{e^x} dx$$

we find that:

$$\begin{aligned}\int_0^\infty \sum_{m=1}^{n} \frac{B_{2m}}{(2m)!} x^{2m-1} \cdot \frac{x^{s-1}}{e^x} dx &= \sum_{m=1}^{n} \frac{B_{2m}}{(2m)!} \int_0^\infty x^{2m-1} \cdot \frac{x^{s-1}}{e^x} dx \\ = \sum_{m=1}^{n} \frac{B_{2m}}{(2m)!} \int_0^\infty \frac{x^{s+2m-2}}{e^x} dx &= \sum_{m=1}^{n} \frac{B_{2m}}{(2m)!} \Gamma(s+2m-1).\end{aligned} \tag{2}$$

Therefore

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + \sum_{m=1}^{n} \frac{B_{2m}}{(2m)!} \frac{\Gamma(s+2m-1)}{\Gamma(s)} + \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} - \sum_{m=1}^{n} \frac{B_{2m}}{(2m)!} x^{2m-1} \right) \frac{x^{s-1}}{e^x} \, \mathrm{d}x.$$

Finally, recognizing that

$$\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} - \sum_{m=1}^{n} \frac{B_{2m}}{(2m)!} x^{2m-1} = \mathcal{O}(x^{2n+1})$$

as $x \to 0$ the region of convergence is also demonstrated. □