POSITIVE INCREMENT

Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1.

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right) \frac{x^{s-1}}{e^x} dx$$

for $Re(s) > 1$.

Proof. Consider the formula

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx$$

true for $Re(s) > 1$, a proof of this relation can be found on our page: "Relation to the Gamma Function 1".

Notice that:

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1} dx = \int_0^\infty \frac{x^{s-1}}{e^x - 1} \cdot \frac{e^x}{e^x} dx = \int_0^\infty \frac{x^{s-1}}{e^x} \cdot \frac{e^x}{e^x - 1} dx$$

and use the fact that $e^x = \frac{e^x - 1}{1 - e^{-x}}$ to obtain:

$$\begin{aligned}
&\int_0^\infty \frac{x^{s-1}}{e^x} \cdot \frac{e^x}{e^x - 1} dx = \int_0^\infty \frac{x^{s-1}}{e^x} \cdot \frac{e^x - 1}{1 - e^{-x}} \cdot \frac{1}{e^x - 1} dx \\
&= \int_0^\infty \frac{x^{s-1}}{e^x} \left[\frac{e^x - 1}{1 - e^{-x}} \cdot \frac{1}{e^x - 1} + \frac{1}{2} - \frac{1}{2} + \frac{1}{x} - \frac{1}{x} \right] dx \\
&= \int_0^\infty \frac{x^{s-1}}{e^x} \left(\frac{e^x - 1}{1 - e^{-x}} \cdot \frac{1}{e^x - 1} - \frac{1}{2} - \frac{1}{x} \right) dx + \frac{1}{2} \int_0^\infty \frac{x^{s-1}}{e^x} dx + \int_0^\infty \frac{x^{s-2}}{e^x} dx.
\end{aligned} \tag{2}$$

Notice now that $\int_0^\infty \frac{x^{s-1}}{e^x}dx = \Gamma(s)$ and $\int_0^\infty \frac{x^{s-2}}{e^x}dx = \Gamma(s-1)$ by definition. Therefore, we have:

$$\begin{aligned}
\zeta(s) &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x-1}dx \\
&= \frac{1}{\Gamma(s)}\left[\int_0^\infty \frac{x^{s-1}}{e^x}\left(\frac{e^x-1}{1-e^{-x}}\cdot\frac{1}{e^x-1}-\frac{1}{2}-\frac{1}{x}\right)dx + \frac{1}{2}\int_0^\infty \frac{x^{s-1}}{e^x}dx + \int_0^\infty \frac{x^{s-2}}{e^x}dx\right] \\
&= \frac{1}{\Gamma(s)}\left[\int_0^\infty \frac{x^{s-1}}{e^x}\left(\frac{e^x-1}{1-e^{-x}}\cdot\frac{1}{e^x-1}-\frac{1}{2}-\frac{1}{x}\right)dx + \frac{1}{2}\Gamma(s)+\Gamma(s-1)\right] \\
&= \frac{1}{2}+\frac{1}{s-1}+\frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x}\left(\frac{e^x-1}{1-e^{-x}}\cdot\frac{1}{e^x-1}-\frac{1}{2}-\frac{1}{x}\right)dx
\end{aligned} \tag{3}$$

where we used the property of the Gamma function: $\frac{\Gamma(s-1)}{\Gamma(s)} = \frac{1}{s-1}$.

Now to obtain the formula we notice that:

$$\frac{e^x-1}{1-e^{-x}}\cdot\frac{1}{e^x-1} = \frac{e^x}{e^x-1}$$

and also

$$\frac{1}{e^x-1}+1 = \frac{1+e^x-1}{e^x-1} = \frac{e^x}{e^x-1}.$$

Therefore

$$\frac{e^x-1}{1-e^{-x}}\cdot\frac{1}{e^x-1} = \frac{1}{e^x-1}+1$$

$$\Downarrow$$

$$\frac{e^x-1}{1-e^{-x}}\cdot\frac{1}{e^x-1}-\frac{1}{2} = \frac{1}{e^x-1}+\frac{1}{2}$$

and we have:

$$\zeta(s) = \frac{1}{2}+\frac{1}{s-1}+\frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x}\left(\frac{1}{e^x-1}-\frac{1}{x}+\frac{1}{2}\right)dx.$$

□