Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1.

$$\zeta(s) = \frac{1}{(1-2^{1-s})\Gamma(s+1)} \int_0^{\infty} \frac{e^x x^s}{(e^x+1)^2} dx$$

for $Re(s) > 0$.

Proof. Start by considering the equation

$$\zeta(s) = \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x+1} dx$$

true for $Re(s) > 0$, you can find a complete proof on our site.

We obtain the result just by integrating by parts:

$$\int_0^{\infty} \frac{x^{s-1}}{e^x+1} dx = \left| \frac{x^s}{s} \frac{1}{e^x+1} \right|_0^{\infty} - \left(- \int_0^{\infty} \frac{x^s}{s} \frac{e^x}{(e^x+1)^2} dx \right) = \int_0^{\infty} \frac{x^s}{s} \frac{e^x}{(e^x+1)^2} dx.$$

Therefore

$$\begin{aligned} \zeta(s) &= \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x+1} dx = \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^{\infty} \frac{x^s}{s} \frac{e^x}{(e^x+1)^2} dx \\ &= \frac{1}{(1-2^{1-s})\Gamma(s+1)} \int_0^{\infty} \frac{e^x x^s}{(e^x+1)^2} dx \end{aligned} \tag{2}$$

where we used the known property of the Gamma function: $s\Gamma(s) = \Gamma(s+1)$. □