POSITIVE
INCREMENT

Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1.

$$\zeta(s) = \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x+1} dx \tag{2}$$

for $Re(s) > 0$.

Proof. Define:

$$I_n := \int_0^\infty (-1)^n x^{s-1} e^{-(n+1)x} dx$$

using the fact that the Gamma function is defined as $\Gamma(s) = \int_0^\infty := y^{s-1}e^{-y}dy$ we have, changing variable to $y = (n+1)x$:

$$\begin{aligned} I_n &= \int_0^\infty (-1)^n x^{s-1} e^{-(n+1)x} dx = (-1)^n \int_0^\infty \left(\frac{y}{n+1}\right)^{s-1} e^{-y} \frac{dy}{n+1} \\ &= \frac{(-1)^n}{(n+1)^s} \int_0^\infty y^{s-1} e^{-y} dy = \frac{(-1)^n}{(n+1)^s} \Gamma(s). \end{aligned} \tag{3}$$

Remember now that, for $Re(s) > 0$:

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \frac{1}{1-2^{1-s}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^s}$$

$$\Downarrow$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^s} = \zeta(s)(1-2^{1-s})$$

proof of this equation can be found on our site.

Therefore:

$$\sum_{n=0}^{\infty} I_n = \Gamma(s) \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^s} = \Gamma(s)\zeta(s)(1 - 2^{1-s}). \tag{4}$$

$$\Downarrow$$

$$\zeta(s) = \frac{1}{(1 - 2^{1-s})\Gamma(s)} \sum_{n=0}^{\infty} I_n$$

so that all that is left to prove is that:

$$\sum_{n=0}^{\infty} I_n = \int_0^{\infty} \frac{x^{s-1}}{e^x + 1} dx.$$

Start by remembering that for $|x| < 1$ we have:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\Downarrow$$

$$\frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$$

$$\Downarrow$$

$$\frac{x}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^{n+1}. \tag{5}$$

In our case we have:

$$\frac{1}{e^x + 1} = \frac{1}{e^x + 1} \cdot \frac{e^{-x}}{e^{-x}} = \frac{e^{-x}}{1 + e^{-x}}$$

and $|e^{-x}| < 1$ for $x > 0$, therefore using 5:

$$\frac{1}{e^x + 1} = \frac{e^{-x}}{1 + e^{-x}} = \sum_{n=0}^{\infty} (-1)^n e^{-(n+1)x}$$

$$\Downarrow$$

$$\frac{x^{s-1}}{e^x + 1} = \sum_{n=0}^{\infty} (-1)^n x^{s-1} e^{-(n+1)x} \tag{6}$$

for $x > 0$.

Now fix $C > 0$ and define:

$$I_{n,C} := \int_C^{\infty} (-1)^n x^{s-1} e^{-(n+1)x} dx$$

it is obvious that: $\lim_{C \to 0} I_{n,C} = I_n$.

Consider that

$$\sum_{n=0}^{\infty} I_{n,C} = \sum_{n=0}^{\infty} \int_{C}^{\infty} (-1)^n x^{s-1} e^{-(n+1)x} dx = \int_{C}^{\infty} \sum_{n=0}^{\infty} (-1)^n x^{s-1} e^{-(n+1)x} dx \quad (7)$$

we are allowed to switch the order of integration and summation using absolute convergence.

For $x \in [C, \infty]$ we have $|e^{-x}| < 1$ and therefore we can use 6 to obtain:

$$\sum_{n=0}^{\infty} I_{n,C} = \int_{C}^{\infty} \sum_{n=0}^{\infty} (-1)^n x^{s-1} e^{-(n+1)x} dx = \int_{C}^{\infty} \frac{x^{s-1}}{e^x + 1} dx.$$

Using now the definition of $I_{n,C}$:

$$\sum_{n=0}^{\infty} I_n = \sum_{n=0}^{\infty} \lim_{C \to 0} I_{n,C} = \lim_{C \to 0} \sum_{n=0}^{\infty} I_{n,C} = \lim_{C \to 0} \int_{C}^{\infty} \frac{x^{s-1}}{e^x + 1} dx = \int_{0}^{\infty} \frac{x^{s-1}}{e^x + 1} dx$$

where we are allowed to exchange the order of limit and sum using absolute convergence. □