Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1.

$$\zeta(s) = \frac{1}{\Gamma(s+1)} \int_0^{\infty} \frac{e^x x^s}{(e^x - 1)^2} dx \tag{2}$$

where $\Gamma(s) := \int_0^{\infty} x^{s-1} e^{-x} dx$ *is the Gamma function and* $Re(s) > 1$.

Proof. First consider the formula

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx$$

this is sometimes known as First Relation to the Gamma Function.

Proceed by integrating by parts, remembering that $Re(s) > 1$:

$$\int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx = \left| \frac{x^s}{s} \frac{1}{e^x - 1} \right|_0^{\infty} - \int_0^{\infty} \frac{x^s}{s} \cdot \frac{-e^x}{(e^x - 1)^2} = \int_0^{\infty} \frac{x^s}{s} \cdot \frac{e^x}{(e^x - 1)^2}$$

therefore

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx = \frac{1}{\Gamma(s)s} \int_0^{\infty} \frac{e^x x^s}{(e^x - 1)^2} = \frac{1}{\Gamma(s+1)} \int_0^{\infty} \frac{e^x x^s}{(e^x - 1)^2} dx$$

where we used the know property of $\Gamma(s)$: $s\Gamma(s) = \Gamma(s+1)$. □