POSITIVE INCREMENT

**Definition 1.** *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

*for* $Re(s) > 1$.

**Theorem 1.**

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \tag{2}$$

*where* $\Gamma(s) := \int_0^{\infty} x^{s-1} e^{-x} dx$ *is the Gamma function and* $Re(s) > 1$.

This is a standard result that can be found in almost any book discussing the Riemann Zeta function ([1],[3], or[2] for example).
The proof we discuss here is a more detailed version of the one contained in [3].

*Proof.* Start by noticing that, for $Re(s) > 0$:

$$\int_0^{\infty} x^{s-1} e^{-nx} dx = \int_0^{\infty} \left(\frac{y}{n}\right)^{s-1} e^{-y} \frac{dy}{n} = \frac{1}{n^s} \int_0^{\infty} y^{s-1} e^{-y} dy$$

now by definition

$$\frac{1}{n^s} \int_0^{\infty} y^{s-1} e^{-y} dy = \frac{\Gamma(s)}{n^s}$$

hence

$$\Gamma(s)\zeta(s) = \sum_{n=1}^{\infty} \frac{\Gamma(s)}{n^s} = \sum_{n=1}^{\infty} \int_0^{\infty} x^{s-1} e^{-nx} dx. \tag{3}$$

See now that, for $Re(s) > 1$, the integral:

$$\sum_{n=1}^{\infty} \int_0^{\infty} x^{Re(s)-1} e^{-nx} dx = \Gamma(Re(s))\zeta(Re(s))$$

converges, we can therefore switch the order of summation and integration in equation 3 by absolute convergence, so that:

$$\Gamma(s)\zeta(s) = \sum_{n=1}^{\infty} \frac{\Gamma(s)}{n^s} = \sum_{n=1}^{\infty} \int_0^{\infty} x^{s-1} e^{-nx} dx$$
$$= \int_0^{\infty} x^{s-1} \sum_{n=1}^{\infty} e^{-nx} dx = \int_0^{\infty} x^{s-1} \left( \sum_{n=0}^{\infty} e^{-nx} - 1 \right) dx = \int_0^{\infty} x^{s-1} \left( \frac{1}{1-e^{-x}} - 1 \right) dx$$
$$= \int_0^{\infty} x^{s-1} \left( \frac{1-1+e^{-x}}{1-e^{-x}} \right) dx = \int_0^{\infty} x^{s-1} \left( \frac{e^{-x}e^{x}}{(1-e^{-x})\, e^{x}} \right) dx = \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} \tag{4}$$

where we used the fact that $\left|\frac{1}{e^x}\right| < 1$ and that $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ for $|q| < 1$.

□

## References


[1] Tom M Apostol. *Introduction to analytic number theory.* Springer Science & Business Media, 2013.

[2] Harold M Edwards. *RiemannÆs zeta function.* Vol. 58. Academic press, 1974.

[3] Edward Charles Titchmarsh and David Rodney Heath-Brown. *The theory of the Riemann zeta-function.* Oxford university press, 1986.