Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1.

$$\zeta(s) = \frac{1}{1-2^{-s}} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s}$$

for $Re(s) > 1$.

Proof. Start by evaluating:

$$(1-2^s)\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} - 2^s \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Splitting the sum that defines the Riemann Zeta function in those terms that have even denominator and those terms that have odd denominator we find:

$$2^s \sum_{n=1}^{\infty} \frac{1}{n^s} = 2^s \left(\sum_{n=1}^{\infty} \frac{1}{(2n)^s} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} \right) = \sum_{n=1}^{\infty} \frac{2^s}{(2n)^s} + 2^s \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s}.$$

Notice now that, by simple computation $\sum_{n=1}^{\infty} \frac{2^s}{(2n)^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} =: \zeta(s)$.

Therefore:

$$(1-2^s)\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} - 2^s \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} - 2^s \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} - \sum_{n=1}^{\infty} \frac{1}{n^s} = -2^s \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s}$$

$$\Downarrow$$

$$\zeta(s) = -\frac{2^s}{1-2^s} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} = \frac{1}{2^{-s}(2^s-1)} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} = \frac{1}{1-2^{-s}} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s}.$$

□