POSITIVE INCREMENT

Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1.

$$\lim_{s \to 1} \left(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} \right) = \gamma \tag{2}$$

where $\gamma := \lim_{n\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \log N \right)$ *is the Euler-Mascheroni constant.*

Proof. Remember the Laurent Expansion of the Riemann Zeta function:

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0} \frac{(-1)^k}{k!} \gamma_k (s-1)^k = \frac{1}{s-1} + \gamma + \mathcal{O}(s-1). \tag{3}$$

Remark 1. *The constants* γ_k *are called Stieltjes constants and are defined as:*

$$\gamma_k := \lim_{n\to\infty} \left(\sum_{n=1}^{N} \frac{(\log n)^k}{n} - \frac{(\log N)^{k+1}}{k+1} \right).$$

It is clear that $\gamma_0 = \gamma$, *by definition.*

Equation 3 implies:

$$\zeta'(s) = -\frac{1}{(s-1)^2} + \mathcal{O}(1)$$

Remark 2. *Differentiation term by term is clearly allowed.*

Hence:

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{-\frac{1}{(s-1)^2} + \mathcal{O}(1)}{\frac{1}{s-1} + \gamma + \mathcal{O}(s-1)}$$

$$\Downarrow$$

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{-\frac{1}{(s-1)} + \mathcal{O}(s-1)}{1 + \gamma(s-1) + \mathcal{O}\left((s-1)^2\right)}. \tag{4}$$

Remember the alternating geometric series:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 \cdots$$

valid when $|x| < 1$. In our case, as $s \approx 1$, $\left|\gamma(s-1) + \mathcal{O}\left((s-1)^2\right)\right| < 1$. Therefore:

$$\frac{1}{1 + \gamma(s-1) + \mathcal{O}\left((s-1)^2\right)} = 1 - \gamma(s-1) + \mathcal{O}\left((s-1)^2\right) + \left(\gamma(s-1) + \mathcal{O}\left((s-1)^2\right)\right)^2 \cdots$$

$$\Downarrow$$

$$\frac{1}{1 + \gamma(s-1) + \mathcal{O}\left((s-1)^2\right)} = 1 - \gamma(s-1) + \mathcal{O}\left((s-1)^2\right).$$

With this in mind, we can write equation 4 as:

$$\frac{\zeta'(s)}{\zeta(s)} = \left(-\frac{1}{(s-1)} + \mathcal{O}(s-1)\right)\left(1 - \gamma(s-1) + \mathcal{O}\left((s-1)^2\right)\right)$$

$$\Downarrow$$

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{(s-1)} + \gamma + \mathcal{O}(s-1)$$

$$\Downarrow$$

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{(s-1)} = \gamma + \mathcal{O}(s-1)$$

$$\Downarrow$$

$$\lim_{s\to 1}\left(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{(s-1)}\right) = \gamma.$$

□