Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1. *The Laurent expansion for* $\zeta(s)$ *around the pole at* $s = 1$ *is:*

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \gamma_k (s-1)^k \tag{2}$$

where γ_k *are called Stieltjes constants and are defined via*

$$\gamma_k = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{(\log(n))^k}{n} - \frac{(\log(N))^{k+1}}{k+1} \right).$$

Proof. We will use the following equation for the Zeta function:

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s \int_N^{\infty} \frac{x - [x]}{x^{s+1}} dx \tag{3}$$

true for $Re(s) > 0$, $N = 1, 2, \cdots$ and $[x] := max\{k \in \mathbb{Z} : k \le x\}$, known as "floor" of x or integer part of x this is sometimes known as Representation by Euler-Maclaurin Formula.

Focus on the term:

$$\sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1}$$

adding and subtracting $\frac{1}{s-1}$ gives us

$$\frac{1}{s-1} + \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{1}{s-1} = \frac{1}{s-1} + \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + \frac{1}{1-s} = \frac{1}{s-1} + \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s} - 1}{1-s}.$$

Write n^{1-s} and N^{1-s} in the exponential form and substitute with their power series about $s = 1$:

$$\begin{aligned}
&\frac{1}{s-1} + \sum_{n=1}^{N} \frac{1}{n} e^{(1-s)\log(n)} - \frac{e^{(1-s)\log(N)} - 1}{1-s} \\
&\quad = \frac{1}{s-1} + \sum_{k=0}^{\infty} \left(\sum_{n=1}^{N} \frac{1}{n} \frac{(1-s)^k \log(n)^k}{k!} \right) - \frac{\sum_{k=0}^{\infty} \frac{(1-s)^k \log(N)^k}{k!} - 1}{(1-s)} \\
&\quad = \frac{1}{s-1} + \sum_{k=0}^{\infty} \left(\sum_{n=1}^{N} \frac{1}{n} \frac{(1-s)^k \log(n)^k}{k!} \right) - \frac{\sum_{k=1}^{\infty} \frac{(1-s)^k \log(N)^k}{k!}}{(1-s)} \\
&\quad = \frac{1}{s-1} + \sum_{k=0}^{\infty} \left(\sum_{n=1}^{N} \frac{1}{n} \frac{(1-s)^k \log(n)^k}{k!} \right) - \sum_{k=0}^{\infty} \frac{(1-s)^{k+1} \log(N)^{k+1}}{(k+1)!(1-s)} \\
&\quad = \frac{1}{s-1} + \sum_{k=0}^{\infty} \left[\sum_{n=1}^{N} \frac{1}{n} \frac{(1-s)^k \log(n)^k}{k!} - \frac{(1-s)^k \log(N)^{k+1}}{(k+1)!} \right] \\
&\quad = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(1-s)^k}{k!} \left[\sum_{n=1}^{N} \frac{\log(n)^k}{n} - \frac{\log(N)^{k+1}}{k+1} \right]
\end{aligned} \tag{4}$$

By using 4 in 3 we find

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(1-s)^k}{k!} \left[\sum_{n=1}^{N} \frac{\log(n)^k}{n} - \frac{\log(N)^{k+1}}{k+1} \right] - s \int_N^{\infty} \frac{x - [x]}{x^{s+1}} dx \tag{5}$$

Given this is true $\forall N \in \mathbb{N}$ it will remain true for $N \to \infty$, therefore:

$$\begin{aligned}
\zeta(s) &= \lim_{N\to\infty} \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(1-s)^k}{k!} \left[\sum_{n=1}^{N} \frac{\log(n)^k}{n} - \frac{\log(N)^{k+1}}{k+1} \right] - s \int_N^{\infty} \frac{x - [x]}{x^{s+1}} dx \\
&= \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(1-s)^k}{k!} \left[\lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{\log(n)^k}{n} - \frac{\log(N)^{k+1}}{k+1} \right) \right] \\
&= \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(1-s)^k}{k!} \gamma_k = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \gamma_k (s-1)^k
\end{aligned} \tag{6}$$

where we used the fact that $|\int_N^{\infty} \frac{x-[x]}{x^{s+1}} dx| \leq \int_N^{\infty} |\frac{x-[x]}{x^{s+1}}| dx \leq \int_N^{\infty} |\frac{1}{x^{s+1}}| dx \leq \int_N^{\infty} \frac{1}{x^{\delta+1}} dx$ for any $0 < \delta \leq Re(s)$ if $Re(s) > 1$ or any $Re(s) \geq \delta > 0$ if $Re(s) < 1$, therefore $\lim_{N\to\infty} -s \int_N^{\infty} \frac{x-[x]}{x^{s+1}} dx = 0$. □