POSITIVE INCREMENT

Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1.

$$\zeta^{(k)}(s) = (-1)^k \sum_{n=2}^{\infty} (\log(n))^k n^{-s} \tag{2}$$

for $Re(s) > 1$, $k = 1, 2, 3, \cdots$.

Proof. Applying repeatedly the fact that the sum function of a Dirichlet Series (like the Riemann Zeta function) is analytic in its half-plane of convergence ($Re(s) > 1$ for $\zeta(s)$) and can be obtained by differentiating term by term (see [1] theorem 11.12, or also our proof for the first derivative), one finds

$$\zeta'(s) = -\sum_{n=2}^{\infty} (log(n)) n^{-s}$$

$$\zeta''(s) = \sum_{n=2}^{\infty} (log(n))^2 n^{-s}$$

$$\zeta'''(s) = -\sum_{n=2}^{\infty} (log(n))^3 n^{-s}$$

and so forth. □

References

[1] Tom M Apostol. *Introduction to analytic number theory.* Springer Science & Business Media, 2013.