Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1.

$$\zeta(s) = \frac{2^{s-1}}{\Gamma(s+1)} \int_0^{\infty} \frac{x^s}{\sinh(x)^2} dx$$

for $Re(s) > 1$.

Proof. Start by considering the equation:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \tag{2}$$

proof of this formula can be found on our site.

Proceed by integrating by parts:

$$\begin{aligned}
&\int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx = \left| \frac{x^s}{s} \cdot \frac{1}{e^x - 1} \right|_0^{\infty} + \int_0^{\infty} \frac{x^s}{s} \cdot \frac{e^x}{(e^x - 1)^2} dx \\
&\text{notice that the first term is zero and change the variable to } y = \frac{x}{2}\text{:} \\
&= 2^s \int_0^{\infty} \frac{y^s}{s} \cdot \frac{e^{2y} \cdot 2dy}{(e^{2y} - 1)^2} = 2^{s-1} \int_0^{\infty} \frac{y^s}{s} \cdot \frac{4e^{2y}}{(e^{2y} - 1)^2} dy \\
&= 2^{s-1} \int_0^{\infty} \frac{y^s}{s} \cdot \left(\frac{2e^y}{(e^{2y} - 1)} \right)^2 dy = \frac{2^{s-1}}{s} \int_0^{\infty} \frac{y^s}{\sinh(y)^2} dy
\end{aligned} \tag{3}$$

where we used the fact that $\sinh(y) = \frac{e^{2y}-1}{2e^y}$.

Going back to equation 2 we conclude by using the known property of the Gamma function $s\Gamma(s) = \Gamma(s+1)$:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx = \frac{2^{s-1}}{s\Gamma(s)} \int_0^\infty \frac{x^s}{\sinh(x)^2} dx = \frac{2^{s-1}}{\Gamma(s+1)} \int_0^\infty \frac{x^s}{\sinh(x)^2} dx.$$

□