POSITIVE INCREMENT

**Definition 1.** *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

*for* $Re(s) > 1$.

**Theorem 1.**

$$\zeta(s) = \frac{2^{s-1}}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x} \cdot csch(x) dx \tag{2}$$

*for* $Re(s) > 1$.

*Proof.* Remember that the hyperbolic cosecant is defined as:

$$\text{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}.$$

Therefore, the integral on the right of equation 2 can be written as:

$$\int_0^\infty \frac{x^{s-1}}{e^x} \cdot \text{csch}(x) dx = \int_0^\infty \frac{x^{s-1}}{e^x} \cdot \frac{2}{e^x - e^{-x}} dx = 2 \int_0^\infty \frac{x^{s-1}}{e^{2x} - 1} dx. \tag{3}$$

Change variable to $y = 2x$ to obtain:

$$2 \int_0^\infty \frac{x^{s-1}}{e^{2x} - 1} dx = 2 \int_0^\infty \left(\frac{y}{2}\right)^{s-1} \cdot \frac{1}{e^y - 1} \frac{dy}{2} = \int_0^\infty \frac{y^{s-1}}{2^{s-1}} \cdot \frac{1}{e^y - 1} dy. \tag{4}$$

Hence:

$$\frac{2^{s-1}}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x} \cdot \text{csch}(x) dx = \frac{2^{s-1}}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{2^{s-1}} \cdot \frac{1}{e^x - 1} dx = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx.$$

The last term of the equation is connected to the Riemann Zeta Function by the Relation to the Gamma Function 1:

$$\frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx = \zeta(s)$$

In conclusion:

$$\zeta(s) = \frac{2^{s-1}}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x} \cdot \operatorname{csch}(x) dx.$$

□