POSITIVE INCREMENT

Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Definition 2. *The Riemann Xi function is defined as*

$$\xi(s) := \frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{2}$$

Theorem 1.

$$\xi(s) = \frac{1}{2}\prod_{\rho}\left(1-\frac{s}{\rho}\right) \tag{3}$$

where the product runs over all ρ *roots of the function* ξ.

Remark 1. *Remember that the roots of the* ξ *function are exactly the* ***Non trivial*** *zeros of the Zeta Function.*

Proof. Using the **Hadamard Factorization Theorem**, we obtain:

$$\xi(s) = \frac{1}{2}e^{Bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}. \tag{4}$$

Remark 2. *We explained rigorously how to obtain this formula in our proof of Hadamard's Factorization Formula for the Zeta function.*

Computing the logarithmic derivative of this factorization, we have:

$$\frac{\xi'(s)}{\xi(s)} = B + \sum_{\rho}\left(\frac{1}{s-\rho_n} + \frac{1}{\rho_n}\right). \tag{5}$$

However, the ξ function satisfies the functional equation:

$$\xi(s) = \xi(1-s). \tag{6}$$

Therefore we have:

$$\frac{\xi'(s)}{\xi(s)} = -\frac{\xi'(1-s)}{\xi(1-s)}.$$

Hence, using equation 5:

$$B + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) = -B - \sum_\rho \left(\frac{1}{1-s-\rho} + \frac{1}{\rho}\right)$$
$$\Downarrow$$
$$2B = -\sum_\rho \left(\frac{1}{1-s-\rho} + \frac{1}{\rho}\right) - \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$
$$\Downarrow$$
$$2B = \sum_\rho -\frac{1}{1-s-\rho} - \frac{1}{\rho} - \frac{1}{s-\rho} - \frac{1}{\rho}$$
$$\Downarrow$$
$$2B = -2\sum_\rho \frac{1}{\rho} - \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{1-s-\rho}\right)$$
$$\Downarrow$$
$$B = -\sum_\rho \frac{1}{\rho} - \frac{1}{2}\sum_\rho \left(\frac{1}{s-\rho} - \frac{1}{s-(1-\rho)}\right).$$

The functional equation 6 also implies that ρ is a zero if and only if $1-\rho$ is a zero, therefore in the last sum each term cancels out and we have:

$$B = -\sum_\rho \frac{1}{\rho}.$$

Which substituted in equation 4 gives us:

$$\begin{aligned}\xi(s) &= \frac{1}{2}e^{-s\sum_\rho \frac{1}{\rho}} \prod_\rho \left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}} = \frac{1}{2}\prod_\rho e^{-\frac{s}{\rho}}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}} \\ &= \frac{1}{2}\prod_\rho \left(1-\frac{s}{\rho}\right)\end{aligned} \tag{7}$$

which is exactly equation 3. □