**Definition 1.** *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

*for* $Re(s) > 1$.

**Theorem 1.**

$$\zeta(s) = \frac{(2\pi)^s e^{-s-\frac{\gamma}{2}s}}{2(s-1)\Gamma\left(\frac{1}{2}s+1\right)} \prod_{p} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}} \tag{2}$$

*where* $\gamma$ *is Euler's constant defined as* $\gamma := \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n)\right)$ *and the product is over all roots* $\rho$ *of* $\zeta$ *with* $Re(\rho) > 0$.

The demonstration is a more detailed version of the one seen in [1].

*Proof.* The main ingredient of this proof is Hadamard's factorization Theorem.

It states that, an entire function $f$ of order $N$ can be written in the **canonical Hadamard's Representation**:

$$f(s) = s^m e^{Q(s)} \prod_{n=1}^{\infty} E_p\left(\frac{s}{a_n}\right).$$

Here $E_p(s)$ are the **Hadamard canonical Factors** defined as:

$$E_p(s) := (1-s) \prod_{k=1}^{p} e^{\frac{s^k}{k}}$$

$a_n$ are the roots of $f$ that are not zero ($a_n \neq 0$), $m$ is the order of the zero of $f$ at $z = 0$ (where if $f(0) \neq 0$ we have $m = 0$), $Q$ is a polynomial of a certain degree $q$ and $p$ is the smallest non-negative integer such that the series

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}}$$

converges.

The value $g$ defined as $g := \max\{p, q\}$ is called genus of the function $f$, note that $p \geq 1$ hence $g \geq 1$, it is also true that $g \leq N \leq g + 1$ therefore if $N$ is an integer either $g = N - 1$ or $g = N$.

We will proceed by applying this theorem to the Riemann $\xi$ function.

**Remark 1.** *For those unfamiliar with this function we recommend our Lesson regarding Riemann's breaktrough article and our Proof of its reflection formula.*

To apply the Theorem, first we need to prove that this is an entire function of order $N = 1$; remember that the $\xi$ function is defined as:

$$\xi(s) := \frac{1}{2}s(s-1)\Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{s}{2}}\zeta(s)$$

therefore the problems with regularity can only be the ones coming from the factors $\zeta(s)$ and $\Gamma\left(\frac{1}{2}s\right)$. Start from the case $Re(s) > 0$:

The $\Gamma$ function converges absolutely for $Re(s) > 0$; you can find rigorous proof of this statement on our site.

The $\zeta$ function can be extended to the whole plane in many different ways, (see for example its First Functional Equation) finding that the only singularity is a simple pole at $s = 1$. On the other hand, in the definition of $\xi$ we have $(s-1)\zeta(s)$ which is regular even in $s = 1$.

Therefore $\xi(s)$ is regular $\forall s \in \mathbb{C}|\ Re(s) > 0$.

Since $\xi(s)$ satisfies the equation:

$$\xi(s) = \xi(1-s) \qquad \text{(see Remark 1)} \tag{3}$$

it is also regular $\forall s \in \mathbb{C}|Re(s) < 1$, hence is an integral function.

Remember now that the *order* of an entire function $f(\cdot)$ is defined as:

$$N := \inf\left\{\alpha > 0|\ f(s) \in \mathcal{O}(e^{|z|^\alpha})\right\}.$$

Once again to prove that the order of $\xi$ is 1 we will focus on the $\Gamma$ and $\zeta$ factors that appear in the definition:

Firstly:

$$\left|\Gamma\left(\frac{1}{2}s\right)\right| = \left|\int_0^\infty t^{\frac{1}{2}s-1}e^{-t}dt\right| \leq \int_0^\infty t^{\frac{1}{2}Re(s)-1}e^{-t}dt = \Gamma\left(\frac{1}{2}Re(s)\right) = \mathcal{O}\left(e^{ARe(s)\ln(Re(s))}\right). \tag{4}$$

While, for the $\zeta$ function, remember that it satisfies the equation:

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s\int_1^\infty \frac{x - [x] - \frac{1}{2}}{x^{s+1}}dx$$

sometimes referred to as Representation by Euler-Maclaurin Formula.

For $Re(s) \geq \frac{1}{2}$ and $|s-1| > A$, we find:

$$\zeta(s) = \mathcal{O}\left(|s| \int_1^\infty \frac{dx}{x^{\frac{3}{2}}}\right) + \mathcal{O}(1) = \mathcal{O}(|s|)$$

and therefore

$$\xi(s) = \mathcal{O}(e^{A|s|\ln(|s|)})$$

for $Re(s) \geq \frac{1}{2}$ and $|s| > A$; using the reflection formula 3 we find that this holds also for $Re(s) \leq \frac{1}{2}$.

Hence by definition $\xi(s)$ is of order at most 1; the order is exactly 1 since as $s \to \infty$ by real values we have $\ln(\zeta(s)) \approx 2^{-s} \Rightarrow \ln(\xi(s)) \approx \frac{1}{2} s \ln(s)$.

We have therefore proven that $\xi$ is an entire function of order $N = 1$, let's apply now Hadamard's factorization Theorem:

$N = 1$ implies either $g = 1$ or $g = 0$, but by definition $g = \max\{p, q\}$ and $p \geq 1$, so $g = 1$ and therefore $p = 1$ and:

$$E_1\left(\frac{s}{\rho}\right) = \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}$$

so that:

$$\xi(s) = s^m e^{Q(s)} \prod_\rho \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}.$$

We know that $\xi(0) = \frac{1}{2}$ and therefore $m = 0$ and we have:

$$\xi(s) = e^{Q(s)} \prod_\rho \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}.$$

We also know that the degree $q$ of $Q$ is at most 1, i.e. $Q(s) = b_o s + c$ where $b_0$ and $c$ are two constants, using again the fact that $\xi(0) = \frac{1}{2}$ we find that:

$$\xi(0) = e^c \prod_\rho \left(1 - \frac{0}{\rho}\right) e^{\frac{0}{\rho}} = e^c = \frac{1}{2}$$

$$\Downarrow$$

$$c = \ln\left(\frac{1}{2}\right)$$

$$\Downarrow$$

$$\xi(s) = \frac{1}{2} e^{b_0 s} \prod_\rho \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}. \quad (5)$$

Some extra work is now necessary to determine $b_0$:

Remember the definition of $\xi(s)$:

$$\xi(s) := \frac{1}{2}s(s-1)\Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{s}{2}}\zeta(s)$$

$$\Downarrow$$

$$\zeta(s) = \frac{2\xi(s)}{s(s-1)\Gamma\left(\frac{1}{2}s\right)}\pi^{\frac{s}{2}}$$

combining this with equation 5 we have:

$$\begin{aligned}\zeta(s) &= \frac{\pi^{\frac{s}{2}}e^{b_0 s}}{s(s-1)\Gamma\left(\frac{1}{2}s\right)}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}} = \frac{e^{bs}}{2\frac{1}{2}s(s-1)\Gamma\left(\frac{1}{2}s\right)}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}} \\ &= \frac{e^{bs}}{2(s-1)\Gamma\left(\frac{1}{2}s+1\right)}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}\end{aligned} \tag{6}$$

where we defined $b$ as $b := b_0 + \frac{1}{2}\ln(\pi)$ and used that $\frac{1}{2}s\Gamma\left(\frac{1}{2}s\right) = \Gamma\left(\frac{1}{2}s+1\right)$.

To calculate $b$ notice that, using the equation to find $\zeta'(s)$, we have:

$$\frac{\zeta'(s)}{\zeta(s)} = b - \frac{1}{s-1} - \frac{1}{2}\cdot\frac{\Gamma'\left(\frac{1}{2}s+1\right)}{\Gamma\left(\frac{1}{2}s+1\right)} + \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)$$

(this calculation is credited to [1]).

Considering the limit as $s \to 0$, this implies:

$$\frac{\zeta'(0)}{\zeta(0)} = b - \frac{1}{-1} - \frac{1}{2}\cdot\frac{\Gamma'(1)}{\Gamma(1)} + \sum_{\rho}\left(\frac{1}{-\rho}+\frac{1}{\rho}\right)$$

$$\Downarrow$$

$$\frac{\zeta'(0)}{\zeta(0)} = b + 1 - \frac{1}{2}\cdot\frac{\Gamma'(1)}{\Gamma(1)}$$

since $\frac{\zeta'(0)}{\zeta(0)} = \ln(2\pi)$ and $\Gamma'(1) = -\gamma$ it follows that $b = \ln(2\pi) - 1 - \frac{1}{2}\gamma$ and therefore using 6:

$$\zeta(s) = \frac{e^{bs}}{2(s-1)\Gamma\left(\frac{1}{2}s+1\right)}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}} = \frac{(2\pi)^s e^{-s-\frac{\gamma}{2}s}}{2(s-1)\Gamma\left(\frac{1}{2}s+1\right)}\prod_{p}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}.$$

□

# References


[1] Edward Charles Titchmarsh and David Rodney Heath-Brown. *The theory of the Riemann zeta-function*. Oxford university press, 1986.