POSITIVE INCREMENT

Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1.

$$\zeta(m+s) = (-1)^{m-1} \frac{\sin(\pi s)}{\pi} \frac{\Gamma(s)}{\Gamma(s+m)} \int_0^{\infty} \psi^{(m)}(1+x) \frac{dx}{x^s} \tag{2}$$

for $0 < Re(s) < 1$ *and* $m = 1, 2, 3, \cdots$.
Where $\psi(x) := \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ *is the Digamma function and* $\psi^{(m)}(x)$ *indicates the* $m-th$ *derivative of the Digamma function.*

Proof. This proof comes mainly from [1], our job here is just to translate and add some details to make it clearer, we will use the common notation $\sigma = Re(s)$.

Let's start by proving that:

$$\frac{1}{k^{s+m}} = \frac{\Gamma(m+1)}{\Gamma(s+m)\Gamma(1-s)} \int_0^{\infty} \frac{dx}{x^s(k+x)^{m+1}} \tag{3}$$

for $m \geq 0$.

Remark 1. *In his article [1] de Bruijn commits an innocuous mistake in enunciating this formula. We give here a correct version with a rigorous proof. This is necessary to obtain the result of Theorem 2.*

We will demonstrate this using induction. First notice that, changing variable to $tk = x$:

$$\int_0^{\infty} \frac{dx}{x^s(x+k)} = \int_0^{\infty} \frac{kdt}{(kt)^s(kt+k)} = \frac{1}{k^s} \int_0^{\infty} \frac{dt}{t^s(t+1)}.$$

Recognize the integral $\int_0^\infty \frac{dt}{t^s(t+1)}$ as a special case of the so called "Beta function", defined as:

$$B(z_1, z_2) := \int_0^\infty \frac{t^{z_1-1}}{(1+t)^{z_1+z_2}} dt.$$

It's a known fact that

$$B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}.$$

Therefore:

$$\int_0^\infty \frac{dx}{x^s(x+k)} = \frac{1}{k^s}\int_0^\infty \frac{dt}{t^s(t+1)} = \frac{1}{k^s}B(1-s, s) = \frac{1}{k^s}\frac{\Gamma(1-s)\Gamma(s)}{\Gamma(1)}.$$

We have proven equation 3 for $m = 0$:

$$\frac{1}{k^s} = \frac{\Gamma(1)}{\Gamma(s)\Gamma(1-s)}\int_0^\infty \frac{dx}{x^s(x+k)}. \tag{4}$$

Now simply differentiate both sides with respect to k to obtain 3 for $m = 1$:

$$\frac{1}{k^s} = \frac{\Gamma(1)}{\Gamma(s)\Gamma(1-s)}\int_0^\infty \frac{dx}{x^s(x+k)}.$$

$$\Downarrow$$

$$\frac{-s}{k^{s+1}} = \frac{\Gamma(1)}{\Gamma(s)\Gamma(1-s)}\int_0^\infty \frac{(-1)dx}{x^s(x+k)^2}$$

$$\Downarrow$$

$$\frac{1}{k^{s+1}} = \frac{1\cdot\Gamma(1)}{s\Gamma(s)\Gamma(1-s)}\int_0^\infty \frac{dx}{x^s(x+k)^2}.$$

$$\Downarrow$$

$$\frac{1}{k^{s+1}} = \frac{\Gamma(2)}{\Gamma(s+1)\Gamma(1-s)}\int_0^\infty \frac{dx}{x^s(x+k)^2}.$$

Where we used Leibniz's Integral rule to differentiate under the integral sign and the known property of the Γ function: $s\Gamma(s) = \Gamma(s+1)$.

We have therefore proven that the formula is true for $m = 1$, assuming it true for m let's prove it now for $m+1$, simply by differentiating again:

$$\frac{1}{k^{s+m}} = \frac{\Gamma(m+1)}{\Gamma(s+m)\Gamma(1-s)}\int_0^\infty \frac{dx}{x^s(k+x)^{m+1}}$$

$$\Downarrow$$

$$\frac{-(s+m)}{k^{s+m+1}} = \frac{\Gamma(m+1)}{\Gamma(s+m)\Gamma(1-s)}\int_0^\infty \frac{-(m+1)dx}{x^s(k+x)^{m+1+1}}$$

$$\Downarrow$$

$$\frac{1}{k^{s+m+1}} = \frac{\Gamma(m+1+1)}{\Gamma(s+m+1)\Gamma(1-s)}\int_0^\infty \frac{dx}{x^s(k+x)^{m+1+1}}$$

Where we used the exact same results as before. This proves 3.

Remember now the expression for the derivatives of the Digamma function (also know as polygamma functions):

$$\psi^{(m)}(x) = (-1)^{m-1} m! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{m+1}}$$

therefore:

$$\psi^{(m)}(1+x) = (-1)^{m-1} m! \sum_{k=0}^{\infty} \frac{1}{(1+x+k)^{m+1}} = (-1)^{m-1} m! \sum_{k=1}^{\infty} \frac{1}{(x+k)^{m+1}}$$

$$\Downarrow$$

$$\sum_{k=1}^{\infty} \frac{1}{(x+k)^{m+1}} = (-1)^{m-1} \frac{\psi^{(m)}(1+x)}{m!}$$

Proceed now to sum each side of equation 3:

$$\sum_{k=1}^{n} \frac{1}{k^{s+m}} = \frac{\Gamma(m+1)}{\Gamma(s+m)\Gamma(1-s)} \int_0^{\infty} \sum_{k=1}^{n} \frac{1}{(k+x)^{m+1}} \frac{dx}{x^s} \tag{5}$$

Therefore, using the fact that, by definition, $\zeta(m+s) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^{s+m}}$, going back to 5 we have, for $0 < \sigma < 1$:

$$\begin{aligned}
\zeta(m+s) &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^{s+m}} = \frac{\Gamma(m+1)}{\Gamma(s+m)\Gamma(1-s)} \int_0^{\infty} \sum_{k=1}^{\infty} \frac{1}{(k+x)^{m+1}} \frac{dx}{x^s} \\
&= \frac{\Gamma(m+1)}{\Gamma(s+m)\Gamma(1-s)} \int_0^{\infty} (-1)^{m-1} \frac{\psi^{(m)}(1+x)}{m!} \frac{dx}{x^s} \\
&= (-1)^{m-1} \frac{\Gamma(m+1)}{m!\Gamma(s+m)\Gamma(1-s)} \int_0^{\infty} \psi^{(m)}(1+x) \frac{dx}{x^s}
\end{aligned} \tag{6}$$

Where we can justify the exchange of limit and integral using Lebesgue's dominated convergence theorem because

$$\left|\sum_{k=1}^{n} \frac{1}{(k+x)^{m+1}}\right| \leq \left|\sum_{k=1}^{\infty} \frac{1}{(k+x)^{m+1}}\right| \leq |\psi^{(m)}(1+x)|$$

.

To obtain formula 2, utilize the Relation to the Sine Function:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \implies \Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \cdot \frac{1}{\Gamma(s)}$$

and the basic property of the Gamma Function $s! = \Gamma(s+1)$:

$$\begin{aligned}
\zeta(m+s) &= (-1)^{m-1} \frac{\Gamma(m+1)}{m!\Gamma(s+m)\Gamma(1-s)} \int_0^{\infty} \psi^{(m)}(1+x) \frac{dx}{x^s} \\
&= (-1)^{m-1} \frac{\Gamma(m+1)}{\Gamma(m+1)\Gamma(s+m)\Gamma(1-s)} \int_0^{\infty} \psi^{(m)}(1+x) \frac{dx}{x^s} \\
&= (-1)^{m-1} \frac{\sin(\pi s)}{\pi} \frac{\Gamma(s)}{\Gamma(s+m)} \int_0^{\infty} \psi^{(m)}(1+x) \frac{dx}{x^s}.
\end{aligned} \tag{7}$$

□

References

[1] NG de Bruijn. "Integralen voor de zeta -functie van Riemann". In: *Mathematica: tijdschrift voor studeerenden voor de acten wiskunde MO en voor studeerenden aan Universiteiten. Afdeeling B* 5 (1937), pp. 170–180.