Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1. *The Riemann zeta function satisfies the equation*

$$\zeta(s) = 2(2\pi)^{s-1} \sin\left(\frac{\pi}{2}s\right) \Gamma(1-s)\zeta(1-s) \tag{2}$$

for $s \neq 0, 1$.

This is a classical result that can be found in many textbooks, the one that follows is an explanation of the proof that can be found in [1].

Proof. We start by proving the following formula for the Γ function:

$$\int_0^{\infty} t^{s-1} \sin(t) dt = \Gamma(s) \sin\left(\frac{\pi}{2}s\right). \tag{3}$$

Notice that

$$\int_0^{\infty} t^{s-1} \sin(t) dt = \int_0^{\infty} t^{s-1} \cdot \frac{e^{it} - e^{-it}}{2i} dt = \frac{1}{2i}\left[\int_0^{\infty} t^{s-1} e^{it} dt - \int_0^{\infty} t^{s-1} e^{-it} dt\right] \tag{4}$$

we can simplify the first integral, using that:

$$\begin{aligned} e^{it} &= 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{n=0}^{\infty} i^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} i^n (-1)^n \frac{(-t)^n}{n!} = \sum_{n=0}^{\infty} i^n \cos(n\pi) \frac{(-t)^n}{n!} \end{aligned} \tag{5}$$

here we are writing $(-1)^n$ as $\cos(n\pi)$.

Hence

$$\int_0^\infty t^{s-1} e^{it} dt = \int_0^\infty t^{s-1} \sum_{n=0}^{\infty} i^n \cos(n\pi) \frac{(-t)^n}{n!} \tag{6}$$

Proceed by applying the so called Ramanujan's Master Theorem, it states that, if a complex valued function $f(t)$ as an expansion of the form:

$$f(t) = \sum_{n=0}^{\infty} \phi(n) \frac{(-t)^n}{n!}$$

then

$$\int_0^\infty t^{s-1} f(t) dt = \Gamma(s)\phi(-s).$$

In our case, where $f(t) = e^{it}$ and $\phi(n) = i^n \cos(n\pi)$, it implies that:

$$\int_0^\infty t^{s-1} e^{it} dt = \Gamma(s) i^{-s} \cos(-s\pi) = \Gamma(s) i^{-s} \cos(s\pi).$$

Using now that $\cos(t) = \frac{e^{it}+e^{-it}}{2}$ and that $\ln(i) = i\frac{\pi}{2}$ we find:

$$\begin{aligned} &\int_0^\infty t^{s-1} e^{it} dt = \Gamma(s) i^{-s} \cos(s\pi) = \Gamma(s) i^{-s} \cdot \frac{e^{is\pi} + e^{-is\pi}}{2} \\ &= \Gamma(s) e^{-s\ln(i)} \cdot \frac{e^{is\pi} + e^{-is\pi}}{2} = \Gamma(s) \cdot \frac{e^{is\pi - is\frac{\pi}{2}} + e^{-is\pi - is\frac{\pi}{2}}}{2} \\ &= \Gamma(s) \cdot \frac{e^{is\frac{\pi}{2}} + e^{-is\frac{3}{2}\pi}}{2} = \Gamma(s) \cdot \frac{e^{is\frac{\pi}{2}} + e^{-is(2\pi - \frac{\pi}{2})}}{2} \\ &= \Gamma(s) \cdot \frac{e^{is\frac{\pi}{2}} + e^{-is(-\frac{\pi}{2})}}{2} = \Gamma(s) \cdot \frac{e^{is\frac{\pi}{2}} + e^{is\frac{\pi}{2}}}{2} = \Gamma(s) e^{is\frac{\pi}{2}}. \end{aligned} \tag{7}$$

Therefore we have:

$$\int_0^\infty t^{s-1} e^{it} dt = \Gamma(s) e^{is\frac{\pi}{2}}.$$

A similar reasoning can be applied to the second integral in equation 4:

$$\begin{aligned} &e^{-it} = 1 - it + \frac{(it)^2}{2!} - \frac{(it)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} = \sum_{n=0}^{\infty} i^n (-1)^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} i^n (-1)^n (-1)^n \frac{(-t)^n}{n!} = \sum_{n=0}^{\infty} i^n \frac{(-t)^n}{n!}. \end{aligned} \tag{8}$$

Hence

$$\int_0^\infty t^{s-1} e^{-it} dt = \Gamma(s) i^{-s} = \Gamma(s) e^{-s\log i} = \Gamma(s) e^{-is\frac{\pi}{2}}. \tag{9}$$

Concluding, equation 4 can be written as:

$$\begin{aligned} &\int_0^\infty t^{s-1} \sin(t) dt = \frac{1}{2i}\left[\int_0^\infty t^{s-1} e^{it} dt - \int_0^\infty t^{s-1} e^{-it} dt\right] = \frac{1}{2i}\left[\Gamma(s) e^{is\frac{\pi}{2}} - \Gamma(s) e^{-is\frac{\pi}{2}}\right] \\ &= \Gamma(s)\left[\frac{e^{is\frac{\pi}{2}} - e^{-is\frac{\pi}{2}}}{2i}\right] = \Gamma(s)\sin\left(\frac{\pi}{2}s\right). \end{aligned} \tag{10}$$

To prove 2 we start by considering the equation:

$$\zeta(s) = s\int_0^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx \tag{11}$$

true for $-1 < Re(s) < 0$, this particular analytical continuation is explained in detail on our site.

It is a known fact that the Fourier series for $[x]-x+\frac{1}{2}$ is:

$$[x]-x+\frac{1}{2} = \sum_{n=1}^{\infty}\frac{\sin(2n\pi x)}{n\pi}$$

valid when x is not an integer, this is a result linked to the fact that $[x]-x+\frac{1}{2}$ is a Bernoulli periodic function.

Substituting in equation 11 we find:

$$\zeta(s) = s\int_0^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx = \frac{s}{\pi}\int_0^\infty \sum_{n=1}^{\infty}\frac{1}{n}\frac{\sin(2n\pi x)}{x^{s+1}}dx = \frac{s}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^\infty \frac{\sin(2n\pi x)}{x^{s+1}}dx. \tag{12}$$

The integration term-by-term is justified as the series converges boundedly, therefore, it is sufficient to prove that:

$$\lim_{\lambda\to\infty}\sum_{n=1}^{\infty}\frac{1}{n}\int_\lambda^\infty \frac{\sin(2n\pi x)}{x^{s+1}}dx = 0$$

for $-1 < Re(s) < 0$.

To do this, simply notice that, integrating by parts:

$$\int_\lambda^\infty \frac{\sin(2n\pi x)}{x^{s+1}}dx = \left|-\frac{\cos(2n\pi x)}{2n\pi x^{s+1}}\right|_\lambda^\infty - \frac{s+1}{2n\pi}\int_\lambda^\infty \frac{\cos(2n\pi x)}{x^{s+2}}dx$$

$$= \mathcal{O}\left(\frac{1}{n\lambda^{Re(s)+1}}\right) + \mathcal{O}\left(\frac{1}{n}\int_\lambda^\infty \frac{dx}{x^{Re(s)+2}}\right) = \mathcal{O}\left(\frac{1}{n\lambda^{Re(s)+1}}\right)$$

and the desired result clearly follows.

Using the change of variables $y = 2n\pi x$, equation 12 becomes:

$$\begin{aligned}\zeta(s) &= s\int_0^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx = \frac{s}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^\infty \frac{\sin(2n\pi x)}{x^{s+1}}dx \\ &= \frac{s}{\pi}\sum_{n=1}^{\infty}\frac{(2n\pi)^s}{n}\int_0^\infty \frac{\sin(y)}{y^{s+1}}dy = \frac{s}{\pi}(2\pi)^s\sum_{n=1}^{\infty}\frac{n^s}{n}\int_0^\infty \frac{\sin(y)}{y^{s+1}}dy \\ &= \frac{s}{\pi}(2\pi)^s\sum_{n=1}^{\infty}\frac{1}{n^{1-s}}\int_0^\infty \frac{\sin(y)}{y^{s+1}}dy = \frac{s}{\pi}(2\pi)^s\zeta(1-s)\int_0^\infty y^{-s-1}\sin(y)dy\end{aligned} \tag{13}$$

the last integral is exactly the one that appears in equation 3, therefore:

$$\zeta(s) = \frac{s}{\pi}(2\pi)^s\zeta(1-s)\Gamma(-s)\sin\left(-\frac{\pi}{2}s\right) = \frac{s}{\pi}(2\pi)^s\zeta(1-s)\{-\Gamma(-s)\}\sin\left(\frac{\pi}{2}s\right)$$
$$= 2^s\pi^{s-1}\zeta(1-s)\{-s\Gamma(-s)\}\sin\left(\frac{\pi}{2}s\right) = 2(2\pi)^{s-1}\sin\left(\frac{\pi}{2}s\right)\Gamma(1-s)\zeta(1-s) \tag{14}$$

in the last passage we used the fact that $s\Gamma(s) = \Gamma(s+1)$.

This proves 2 for $-1 < Re(s) < 0$, however the right-hand side is analytic for all values of s such that $Re(s) < 0$ (due to the definitions of ζ and Γ) so that it provides the analytic continuation of $\zeta(s)$ over the remainder of the plane, this also implies that there are no singularities other than the pole at $s = 1$. □

References

[1] Edward Charles Titchmarsh and David Rodney Heath-Brown. *The theory of the Riemann zeta-function*. Oxford university press, 1986.