Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1.

$$\zeta(s) = \frac{1}{(s-1)} + \frac{\sin(\pi s)}{\pi} \int_0^{\infty} x^{-s} \left(\ln(1+x) - \psi(1+x)\right) dx \tag{2}$$

for $0 < Re(s) < 1$, *where* $\psi(x) := \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ *is the Digamma function.*

This formula was firstly published in [1], we proceed to translate this original proof and add some necessary details, we will use notation $\sigma = Re(s)$.

Proof. Start by defining, for $n \in \mathbb{N}, n \geq 2$

$$C_n(s) = \sum_{k=1}^{n-1} \frac{1}{k^s} - \int_1^n \frac{dt}{t^s},$$

defined for $\sigma > 0$.

Now notice that:

$$\int_0^{\infty} y^{s-1} e^{-ay} dy = \int_0^{\infty} \left(\frac{x}{a}\right)^{s-1} e^{-a\frac{x}{n}} \frac{dx}{a} = \frac{1}{a^s} \int_0^{\infty} x^{s-1} e^{-x} dx = \frac{\Gamma(s)}{a^s}$$

here we used the defintion of the Gamma function: $\Gamma(s) := \int_0^{\infty} x^{s-1} e^{-x} dx$, therefore:

$$\frac{1}{a^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} y^{s-1} e^{-ay} dy \tag{3}$$

for $\sigma > 0$ and $a > 0$.

In the definition of $C_n(s)$ we can use 3 to rewrite $\frac{1}{t^s}$ as $\frac{1}{\Gamma(s)}\int_0^\infty y^{s-1}e^{-ty}dy$:

$$\begin{aligned}\int_1^n \frac{dt}{t^s} &= \frac{1}{\Gamma(s)}\int_1^n\left(\int_0^\infty y^{s-1}e^{-ty}dy\right)dt = \frac{1}{\Gamma(s)}\int_0^\infty y^{s-1}\left(\int_1^n e^{-ty}dt\right)dy \\ &= \frac{1}{\Gamma(s)}\int_0^\infty y^{s-1}\cdot\frac{e^{-y}-e^{-ny}}{y}dy.\end{aligned} \tag{4}$$

Notice that the exchange of integration order is allowed as the integral $\int_0^\infty e^{-ty}y^{s-1}dy$ converges uniformly for $0 < t \leq n$ while the integrand is a continuous function of t.

Equation 3 can also be used on $\sum_{k=1}^{n-1}\frac{1}{k^s}$:

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{1}{k^s} &= \sum_{k=1}^{n-1}\frac{1}{\Gamma(s)}\int_0^\infty y^{s-1}e^{-ky} = \frac{1}{\Gamma(s)}\int_0^\infty\left(\sum_{k=1}^{n-1}e^{-ky}\right)y^{s-1}dy \\ &= \frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{1}{e^y}+\frac{1}{e^{2y}}+\cdots+\frac{1}{e^{(n-1)y}}\right)y^{s-1}dy \\ &= \frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{e^{(n-2)y}+e^{(n-3)y}+\cdots+e^y+1}{e^{(n-1)y}}\right)y^{s-1}dy \\ &= \frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{e^{(n-2)y}+e^{(n-3)y}+\cdots+e^y+1}{e^{(n-1)y}}\right)\cdot\left(\frac{1-e^{-y}}{1-e^{-y}}\right)y^{s-1}dy \\ &= \frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{e^{(n-2)y}+e^{(n-3)y}+\cdots+e^y+1-e^{(n-3)y}-e^{(n-4)y}\cdots-1-e^{-y}}{e^{(n-1)y}-e^{(n-2)y}}\right)y^{s-1}dy \\ &= \frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{e^{(n-2)y}-e^{-y}}{e^{(n-1)y}-e^{(n-2)y}}\right)y^{s-1}dy = \frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{e^{(n-1)y}(e^{-y}-e^{-ny})}{e^{(n-1)y}(1-e^{-y})}\right)y^{s-1}dy \\ &= \frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{e^{-y}-e^{-ny}}{1-e^{-y}}\right)y^{s-1}dy.\end{aligned} \tag{5}$$

Therefore we have:

$$\begin{aligned}C_n(s) &= \sum_{k=1}^{n-1}\frac{1}{k^s}-\int_1^n\frac{dt}{t^s} = \frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{e^{-y}-e^{-ny}}{1-e^{-y}}\right)y^{s-1}dy - \frac{1}{\Gamma(s)}\int_0^\infty y^{s-1}\cdot\frac{e^{-y}-e^{-ny}}{y}dy \\ &= \frac{1}{\Gamma(s)}\int_0^\infty y^{s-1}\left(\frac{e^{-y}-e^{-ny}}{1-e^{-y}}-\frac{e^{-y}-e^{-ny}}{y}\right)dy = \frac{1}{\Gamma(s)}\int_0^\infty y^{s-1}\left(\frac{e^{-y}-e^{-ny}}{1-e^{-y}}\cdot\frac{e^y}{e^y}-\frac{e^{-y}-e^{-ny}}{y}\right)dy \\ &= \frac{1}{\Gamma(s)}\int_0^\infty y^{s-1}\left(\frac{1-e^{-(n-1)y}}{e^y-1}-\frac{e^{-y}-e^{-ny}}{y}\right)dy\end{aligned} \tag{6}$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty y^{s-1} \left(\frac{1}{e^y - 1} - \frac{e^{-y}}{y} \right) dy - \frac{1}{\Gamma(s)} \int_0^\infty y^{s-1} \left(\frac{e^{-(n-1)y}}{e^y - 1} - \frac{e^{-ny}}{y} \right) dy$$
$$= \frac{1}{\Gamma(s)} \int_0^\infty y^{s-1} \left(\frac{1}{e^y - 1} - \frac{e^{-y}}{y} \right) dy - \frac{1}{\Gamma(s)} \int_0^\infty y^{s-1} \left(\frac{e^{-(n-1)y}}{e^y - 1} \cdot \frac{e^{-y}}{e^{-y}} - \frac{e^{-ny}}{y} \right) dy$$
$$= \frac{1}{\Gamma(s)} \int_0^\infty y^{s-1} \left(\frac{1}{e^y - 1} - \frac{e^{-y}}{y} \right) dy - \frac{1}{\Gamma(s)} \int_0^\infty y^{s-1} \left(\frac{e^{-ny}}{1 - e^{-y}} - \frac{e^{-ny}}{y} \right) dy$$
$$= \frac{1}{\Gamma(s)} \int_0^\infty y^{s-1} \left(\frac{1}{e^y - 1} - \frac{e^{-y}}{y} \right) dy - \frac{1}{\Gamma(s)} \int_0^\infty y^{s-1} e^{-ny} \left(\frac{1}{1 - e^{-y}} - \frac{1}{y} \right) dy. \tag{7}$$

The second term $\left| \frac{1}{1-e^{-y}} - \frac{1}{y} \right| = \left| \frac{e^y}{e^y-1} - \frac{1}{y} \right|$ for $y \geq 0$ has an upper bound A, the integral is therefore smaller than

$$A \int_0^\infty e^{-ny} y^{\sigma-1} dy = A \int_0^\infty e^{-x} \left(\frac{x}{n} \right)^{\sigma-1} \frac{dx}{n} = \frac{A}{n^\sigma} \Gamma(\sigma).$$

So the second term in the sum is estimated by $\frac{\Gamma(\sigma)}{|\Gamma(s)|} \frac{A}{n^\sigma}$, but the term $\frac{\Gamma(\sigma)}{|\Gamma(s)|}$ is bounded upwards in any bounded region to the right of the straight line $\sigma = \delta > 0$. In this region, due to this argument and equation 6, we have uniform convergence.

Finally, we have proven that, for every $n \in \mathbb{N}, n \geq 2$, $C_n(s)$ is analytical where $\sigma > 0$; this implies that also $C(s) := \lim_{n\to\infty} C_n(s)$ is analytical for $\sigma > 0$.

By definition of $C_n(s)$:

$$C(s) := \lim_{n\to\infty} C_n(s) = \lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{1}{k^s} - \int_1^n \frac{dt}{t^s} = \sum_{k=1}^{\infty} \frac{1}{k^s} - \int_1^\infty \frac{dt}{t^s} = \zeta(s) - \frac{1}{s-1} \tag{8}$$

for $\sigma > 1$. Having proven that $C(s)$ is analytical for $\sigma > 0$, it follows that the function can be continued analytically to $\sigma > 0$, where: $\zeta(s) - \frac{1}{s-1} = C(s)$.

Notice now that, using 6

$$C(n) = \lim_{n\to\infty} C_n(s) = \frac{1}{\Gamma(s)} \int_0^\infty y^{s-1} \left(\frac{1}{e^y - 1} - \frac{e^{-y}}{y} \right) dy$$

and we have obtained:

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{\Gamma(s)} \int_0^\infty y^{s-1} \left(\frac{1}{e^y - 1} - \frac{e^{-y}}{y} \right) dy. \tag{9}$$

Focus on the term y^{s-1}, using again 3 we have:

$$y^{s-1} = \frac{1}{y^{1-s}} = \frac{1}{\Gamma(1-s)} \int_0^\infty e^{-xy} \frac{1}{x^s} dx$$

which once substituted in equation 9 gives us:

$$\zeta(s) - \frac{1}{s-1} = \frac{1}{\Gamma(s)} \frac{1}{\Gamma(1-s)} \int_0^\infty \left(\int_0^\infty e^{-xy} \frac{dx}{x^s} \right) \left(\frac{1}{e^y - 1} - \frac{e^{-y}}{y} \right) dy. \tag{10}$$

Notice that the integrals can be switched using the same argument used earlier and remember that: $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$, therefore we obtain:

$$\begin{aligned}
&= \frac{\sin(\pi s)}{\pi} \int_0^\infty \left(\int_0^\infty e^{-xy} \frac{dx}{x^s} \right) \left(\frac{1}{e^y - 1} - \frac{e^{-y}}{y} \right) dy = \frac{\sin(\pi s)}{\pi} \int_0^\infty \left(\int_0^\infty e^{-xy} \frac{dx}{x^s} \right) \left(\frac{1}{e^y - 1} \cdot \frac{e^{-y}}{e^{-y}} - \frac{e^{-y}}{y} \right) dy \\
&= \frac{\sin(\pi s)}{\pi} \int_0^\infty \frac{dx}{x^s} \int_0^\infty \left(\frac{e^{-y(1+x)}}{1 - e^{-y}} - \frac{e^{-y(1+x)}}{y} \right) dy \\
&= \frac{\sin(\pi s)}{\pi} \int_0^\infty \frac{dx}{x^s} \int_0^\infty \left(\frac{e^{-y(1+x)}}{1 - e^{-y}} - \frac{e^{-y}}{y} + \frac{e^{-y}}{y} - \frac{e^{-y(1+x)}}{y} \right) dy \\
&= \frac{\sin(\pi s)}{\pi} \int_0^\infty \frac{1}{x^s} \left(\int_0^\infty \frac{e^{-y(1+x)}}{1 - e^{-y}} - \frac{e^{-y}}{y} dy + \int_0^\infty \frac{e^{-y} - e^{-y(1+x)}}{y} dy \right) dx.
\end{aligned} \tag{11}$$

Remember now the integral expression for $\psi(x)$ and $\ln(1+x)$:

$$\psi(x) = \int_0^\infty \frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}} dt, \quad \ln(1+x) = \int_0^\infty \frac{e^{-y} - e^{-y(1+x)}}{y} dy$$

so that 11 implies:

$$\zeta(s) - \frac{1}{s-1} = \frac{\sin(\pi s)}{\pi} \int_0^\infty x^{-s} \left(\ln(1+x) - \psi(1+x) \right) dx.$$

□

References

[1] NG de Bruijn. "Integralen voor de zeta -functie van Riemann". In: *Mathematica: tijdschrift voor studeerenden voor de acten wiskunde MO en voor studeerenden aan Universiteiten. Afdeeling B* 5 (1937), pp. 170–180.