Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1.

$$\zeta'(s) = -\sum_{n=2}^{\infty} (\log(n)) n^{-s} \tag{2}$$

for $Re(s) > 1$.

In [1] one can find a more general proof of this property regarding Dirichlet Series; the interesting part is to demonstrate that the sum function of a Dirichlet Series (like the Riemann Zeta function) is analytic on its half-plane of convergence ($Re(s) > 1$ for $\zeta(s)$) and its derivative is obtained by differentiating term by term. Once this has been proven it is simply a matter of computing the derivative of each term.

We will adapt a simplified version of "Theorem 11.11" of Apostol's book.
The aim of this read is to give quick references for those interested in understanding this formula without necessarily having in depth knowledge of analytic number theory or Dirichlet Series, for this reason some details are left without proof. For those interested in the precise demonstration, we suggest chapter 11 of [1].

Proof. We will only demonstrate that $\zeta(s)$ converges uniformly on every compact subset lying interior to the half-plane $Re(s) > 1$ as Lemma 3 of chapter 11 from [1] explains, this is enough for the function to be analytic where $Re(s) > 1$ and derivable term by term.

It suffices to show that $\zeta(s)$ converges uniformly on every compact rectangle $R = [\alpha, \beta] \times [c, d]$ with $\alpha > 1$.

To do this we will use the estimate obtained in Lemma 2 chapter 11 of [1], that in the case of the Riemann Zeta function becomes:

$$\left|\sum_{a<n\le b}\frac{1}{n^s}\right| \le 2Ma^{Re(s)-Re(s_0)}\left(1+\frac{|s-s_0|}{Re(s)-Re(s_0)}\right)$$

where s_0 is any point in the half plane $Re(s) > 1$, s is any point with $Re(s) > Re(s_0)$ and M is a constant that bounds the partial sums i.e. $\left|\sum_{n\le x}\frac{1}{n^s}\right| \le M$ for all $x \ge 1$ (an example of such a constant can be found by estimating using the generalized Bernoulli inequality).
Choose $s_0 \in \mathbb{R}$ where $1 < s_0 < \alpha$, then if $s \in \mathbb{R}$ we have $Re(s) - Re(s_0) = s - s_0 \ge \alpha - s_0$ and $|s - s_0| < C$ where C is a constant depending on s_0 and R but not on s (one can even choose $C = \beta - 1$).

We have:

$$\begin{aligned}\left|\sum_{a<n\le b}\frac{1}{n^s}\right| &\le 2Ma^{Re(s)-Re(s_0)}\left(1+\frac{|s-s_0|}{Re(s)-Re(s_0)}\right)\\ &\le 2Ma^{s_0-\alpha}\left(1+\frac{C}{\alpha-s_0}\right) = Ba^{s_0-\alpha}\end{aligned} \tag{3}$$

where B is independent of s, $\left(B = 2M\left(1+\frac{C}{\alpha-s_0}\right)\right)$; since $a^{s_0-\alpha} \to 0$ as $a \to \infty$ the Cauchy condition for uniform convergence is satisfied.

Once understood that $\zeta'(s)$ is obtained differentiating term by term one can simply notice that

$$\frac{d}{ds}\frac{1}{n^s} = -\frac{log(n)}{n^s}$$

and use that $log(1) = 0$ to obtain 2. □

References

[1] Tom M Apostol. *Introduction to analytic number theory.* Springer Science & Business Media, 2013.