Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1.

$$\zeta(s) = \prod_{p} \frac{1}{1 - p^{-s}} \tag{2}$$

where the product runs over all prime numbers p *and* $Re(s) > 1$.

This is a very classic result, for a more advanced and general proof see [1].

Proof. Notice first that:

$$\prod_{p} \frac{1}{1 - p^{-s}} = \prod_{p} \frac{1}{1 - \frac{1}{p^s}}$$

now remember that for $|q| < 1$ the geometric series converges i.e. $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$ and notice that for $Re(s) > 1$, $\left|\frac{1}{p^s}\right| < 1$ for any prime p, therefore:

$$\prod_{p} \frac{1}{1 - p^{-s}} = \prod_{p} \frac{1}{1 - \frac{1}{p^s}} = \prod_{p} \sum_{k=0}^{\infty} \left(\frac{1}{p^s}\right)^k$$

now it is simply a matter of noticing that multiplying over all prime numbers p and over all their possible powers means exactly summing over all numbers:

$$\prod_{p} \sum_{k=0}^{\infty} \left(\frac{1}{p^s}\right)^k = 1 + \left(\frac{1}{p_1^1}\right)^s + \left(\frac{1}{p_1^2}\right)^s + \cdots + \left(\frac{1}{p_1^1 * p_2^1}\right)^s + \left(\frac{1}{p_1^1 * p_2^2}\right)^s + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$\square$

References

[1] Tom M Apostol. *Introduction to analytic number theory.* Springer Science & Business Media, 2013.