Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1.

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \frac{\eta(s)}{1-2^{1-s}}$$

for $Re(s) > 0$.
The series on the right is the so called "Dirichlet Eta function":

$$\eta(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}.$$

This is a classic result that can be found in many texts, we discuss a more detailed version of the proof that appears in [1].

Proof. First assume that $Re(s) > 1$, then we have:

$$(1-2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} - \frac{1}{2^{s-1}} \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \frac{2}{2^s} \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} - 2 \sum_{n=1}^{\infty} \frac{1}{(2n)^s}$$

$$= (1 + 2^{-s} + 3^{-s} + \cdots) - 2(2^{-s} + 4^{-s} + 6^{-s} + \cdots)$$

$$= 1 - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}.$$

Which proves the equation for $Re(s) > 1$.

If $Re(s) > 0$ the series on the right converges, (this is a corollary of a more general result on Dirichlet Series, it's explained in detail in "Introduction to Analytic Number Theory" by Tom M. Apostol, page 232, Lemma 2 [1]) so by analytic continuation, the equation also holds for $Re(s) > 0$. □

Corollary 1.
$\zeta(s) < 0$ if $s \in \mathbb{R}$ and $0 < s < 1$.

Proof. When $s \in \mathbb{R}$ the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$ is an alternating series with a positive sum (even better, in [2] one can find a proof that the function is monotonic for $s \in \mathbb{R}$), if $0 < s < 1$ then $(1 - 2^{1-s})$ is negative hence $\zeta(s)$ is also negative. □

References

[1] Tom M Apostol. *Introduction to analytic number theory.* Springer Science & Business Media, 2013.

[2] Jan van de Lune. *Some Inequalities Involving Riemann's Zeta-function.* Mathematisch Centrum (Amsterdam, Netherlands), 1975.