Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1. *The Riemann Zeta function satisfies the equation:*

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{-\infty}^{(0,+)} \frac{t^{s-1}}{e^{-t}-1} dt \tag{2}$$

For $s \neq 1, 2, 3, \cdots$.
Here the integration contour is C*, a loop around the negative real axis; it starts at* $-\infty$*, encircles the origin once in the positive direction without enclosing any of the points* $t = \pm 2\pi i, \pm 4\pi i, \cdots$*, and returns to* $-\infty$.
t^{-s} *has its principal value where* t *crosses the positive real axis and is continuous.*

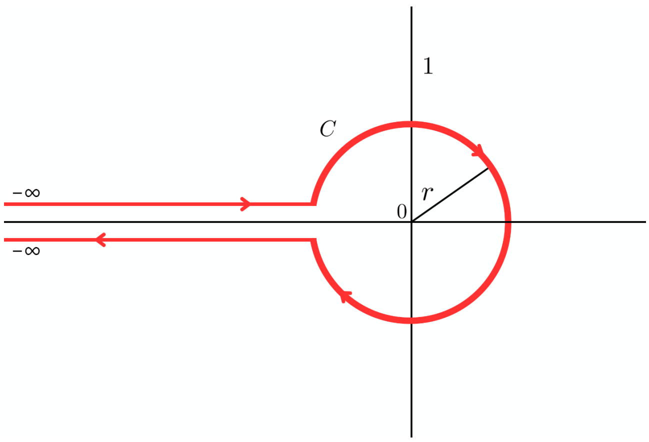

Figure 1: The Contour C

This is a result that is hard to find proven in detail, this is a more complete version of the demonstration from [1].

Proof. Start by considering the formula for the Gamma function known as the Hankel's Loop Integral representation:

$$\frac{1}{\Gamma(s)} = \frac{1}{2\pi i}\int_C e^z z^{-s} dz$$

which implies:

$$\frac{2\pi i}{\Gamma(s)} = \int_C e^z z^{-s} dz \tag{3}$$

the contour C is still the one in the picture above.

Fix $v \in \mathbb{C}$ with $Re(v) > 0$ and substitute $z = vt$ in the integral to obtain:

$$\frac{2\pi i}{\Gamma(s)} = \int_C e^z z^{-s} dz = \int_C e^{vt}(vt)^{-s} v dt = v^{1-s}\int_C e^{vt} t^{-s} dt$$

$$\Downarrow$$

$$\frac{2\pi i}{\Gamma(s)} v^{s-1} = \int_C e^{vt} t^{-s} dt$$

note that, using Cauchy's Theorem, having chosen $v \in \mathbb{C}$ with $Re(v) > 0$, the contour can be left unchanged.

Replacing s with $1 - s$ we have:

$$\frac{2\pi i}{\Gamma(1-s)} v^{1-s-1} = \int_C e^{vt} t^{-(1-s)} dt$$

$$\Downarrow$$

$$v^{-s} = \frac{\Gamma(1-s)}{2\pi i}\int_C e^{vt} t^{s-1} dt$$

substitute v with $v + n$, where $n \in \mathbb{N}_{\geq 0}$, so that we still have $Re(v + n) > 0$:

$$(v+n)^{-s} = \frac{\Gamma(1-s)}{2\pi i}\int_C e^{(v+n)t} t^{s-1} dt.$$

Fix now $z \in \mathbb{C}$ with $|z| \leq 1$.
Restrict the contour C so that any $t \in C$ satisfies $|ze^t| < r < 1$, where r is the radius of the loop around the origin (see Figure 1), once again Cauchy's Theorem ensures that the result does not change. Multiply both sides by z^n:

$$z^n (v+n)^{-s} = z^n \frac{\Gamma(1-s)}{2\pi i}\int_C e^{(v+n)t} t^{s-1} dt$$

taking the sum from zero to infinity we have:

$$\sum_{n=0}^{\infty} (v+n)^{-s} z^n = \frac{\Gamma(1-s)}{2\pi i}\sum_{n=0}^{\infty} z^n \int_C e^{(v+n)t} t^{s-1} dt$$

$$\Downarrow$$

$$\sum_{n=0}^{\infty} (v+n)^{-s} z^n = \frac{\Gamma(1-s)}{2\pi i}\int_C t^{s-1} e^{vt} \sum_{n=0}^{\infty} (ze^t)^n dt$$

we are allowed to move the summation under the integral sign because $|ze^t| < r < 1$.
The series on the right hand side is of the geometric kind and $|ze^t| < 1$, therefore we can use the formula: $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$.

We find:

$$\sum_{n=0}^{\infty}(v+n)^{-s}z^n = \frac{\Gamma(1-s)}{2\pi i}\int_C t^{s-1}e^{vt}(1-ze^t)^{-1}dt. \tag{4}$$

Remark 1. *The series $\sum_{n=0}^{\infty}(v+n)^{-s}z^n$ is sometimes referred to as the function $\Phi(z,s,v)$ (see for example [1]).*

Choosing now $z=1$ and $v=1$ we have:

$$\sum_{n=0}^{\infty}(1+n)^{-s}(1)^n = \frac{\Gamma(1-s)}{2\pi i}\int_C t^{s-1}e^t(1-e^t)^{-1}dt \tag{5}$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty}\frac{1}{n^s} = \frac{\Gamma(1-s)}{2\pi i}\int_C \frac{t^{s-1}}{(e^{-t}-1)}dt \tag{6}$$

$$\Downarrow$$

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i}\int_C \frac{t^{s-1}}{(e^{-t}-1)}dt.$$

□

Corollary 1.

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i(1-2^{1-s})}\int_{-\infty}^{(0,+)}\frac{t^{s-1}}{e^{-t}+1}dt \tag{7}$$

For $s \neq 1,2,3,\cdots$.

Proof. Start again by equation 4, this time choose $z=-1$ and $v=1$ to obtain:

$$\sum_{n=0}^{\infty}(1+n)^{-s}(-1)^n = \frac{\Gamma(1-s)}{2\pi i}\int_C t^{s-1}e^t(1+e^t)^{-1}dt \tag{8}$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} = \frac{\Gamma(1-s)}{2\pi i}\int_C \frac{t^{s-1}}{(e^{-t}+1)}dt. \tag{9}$$

Use now the formula:

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \zeta(s)(1-2^{1-s})$$

proof of this equation can be found on our site.

This implies:

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i(1-2^{1-s})} \int_C \frac{t^{s-1}}{(e^{-t}+1)} dt$$

which is exactly 7. □

References

[1] Arthur Erdélyi. "Higher transcendental functions". In: *Higher transcendental functions* (1953), p. 59.