POSITIVE INCREMENT

Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1. *If s is a complex number with $Re(s) > 1$, we have:*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} \tag{2}$$

*where $\mu(n)$ is the **Möbius function**, defined as:*

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes} \\ 0 & \text{if } n \text{ is divisible by a square} > 1. \end{cases} \tag{3}$$

Proof. This proof requires the use of the Euler product for the Riemann zeta function, that is:

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} \tag{4}$$

where the product runs over all prime numbers p and $Re(s) > 1$. The proof of this classical result can be found on our site.

Equation 4 implies that:

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - p^{-s}\right) = \prod_p \left(1 - \frac{1}{p^s}\right) = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\cdots$$

Let's compute this product step by step:

Firstly, infinite times the product of 1 gives us 1:

$$\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\cdots = 1 + \cdots$$

then we have the product of 1 times a negative fraction of a prime times infinite times 1, which leaves us only with the fraction; this happens for every prime (included 2) and therefore we have:

$$\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\cdots = 1+\sum_p \frac{-1}{p^s}+\cdots.$$

We then find the same product, except this time with two negative fraction of primes; once again this happens for every prime, hence:

$$\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\cdots = 1+\sum_p \frac{-1}{p^s}+\sum_{n=p_1p_2}\frac{-1}{p_1^s}\frac{-1}{p_2^s}+\cdots.$$

Iterating this process we have:

$$\prod_p\left(1-\frac{1}{p^s}\right) = 1+\sum_p \frac{-1}{p^s}+\sum_{n=p_1p_2}\frac{-1}{p_1^s}\frac{-1}{p_2^s}+\sum_{n=p_1p_2p_3}\frac{-1}{p_1^s}\frac{-1}{p_2^s}\frac{-1}{p_3^s}+\cdots$$

but $\frac{-1}{p_1^s}\frac{-1}{p_2^s}=\frac{1}{n^s}$ and $\frac{-1}{p_1^s}\frac{-1}{p_2^s}\frac{-1}{p_3^s}=\frac{-1}{n^s}$ therefore:

$$\prod_p\left(1-\frac{1}{p^s}\right) = 1+\sum_{n=p} \frac{-1}{n^s}+\sum_{n=p_1p_2}\frac{1}{n^s}+\sum_{n=p_1p_2p_3}\frac{-1}{n^s}+\cdots \tag{5}$$

which is exactly

$$\prod_p\left(1-\frac{1}{p^s}\right) = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}. \tag{6}$$

Notice that when n is divisible by a squared prime, it doesn't appear in 5, indeed in the series 6 those numbers have coefficient 0.

□

This equation is just the beginning of the connection between these two topics. If you want to know how the Möbius Function can lead to an equivalent formulation of the Riemann Hypothesis check out our complete lesson!