Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1. *The Riemann Zeta function satisfies the equation:*

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} \tag{2}$$

for $Re(s) > 1$. *Here* $\phi(n)$ *denotes the* ***Euler Totient Function***. $\phi(n)$ *counts the positive integers up to* n *and coprime to* n. *For example:*

$$\phi(1) = 1, \quad \phi(2) = 1, \quad \phi(3) = 2, \quad \phi(11) = 10, \quad \phi(24) = 8, \quad \phi(35) = 24, \quad \phi(37) = 36.$$

Proof. We start by proving that:

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d} \tag{3}$$

where $\mu(n)$ denotes the **Möbius Function** defined as:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1; \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes;} \\ 0 & \text{if } n \text{ is divisible by a square} > 1. \end{cases} \tag{4}$$

we add some details to the demonstration that can be found in [1]:

Rewrite the definition of $\phi(n)$ as

$$\phi(n) = \sum_{k=1}^{n} \left[\frac{1}{(n,k)} \right] \tag{5}$$

where (n,k) denotes the **Greatest Common Denominator** between n and k. It is a known fact that:

$$\sum_{d|n} \mu(d) = \left[\frac{1}{n}\right]$$

(see [1] for the proof.) Therefore Equation 5 can be written as:

$$\phi(n) = \sum_{k=1}^{n} \sum_{d|(n,k)} \mu(d) = \sum_{k=1}^{n} \sum_{\substack{d|n \\ d|k}} \mu(d).$$

Summing over all d that divide both n and k means, for every divisor d of n, summing over all those k in the range $1 \leq k \leq n$ which are multiples of d. If we write $k = qd$ then $1 \leq k \leq n$ if and only if $1 \leq q \leq \frac{n}{d}$. Hence we conclude:

$$\phi(n) = \sum_{k=1}^{n} \sum_{\substack{d|n \\ d|k}} \mu(d) = \sum_{d|n} \sum_{q=1}^{\frac{n}{d}} \mu(d) = \sum_{d|n} \mu(d) \sum_{q=1}^{\frac{n}{d}} 1 = \sum_{d|n} \mu(d) \frac{n}{d}.$$

We will use Equation 3 to obtain our main formula:

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} \frac{n}{d} \mu(d) = \sum_{n=1}^{\infty} \frac{1}{n^{s-1}} \sum_{d|n} \frac{\mu(d)}{d}. \tag{6}$$

Call $n = dk$ and notice that $d|n$ if and only if $n = dk$, therefore summing over all n is equivalent to summing over all possible d and k, that is to say:

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^{s-1}} \sum_{d|n} \frac{\mu(d)}{d} = \sum_{d=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(dk)^{s-1}} \frac{\mu(d)}{d} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^s} \sum_{k=1}^{\infty} \frac{1}{k^{s-1}}. \tag{7}$$

By definition, for $Re(s) > 1$ we have:

$$\sum_{k=1}^{\infty} \frac{1}{k^{s-1}} = \zeta(s-1)$$

also, it is true that:

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d^s} = \frac{1}{\zeta(s)}$$

a demonstration of this equation can be found on our site.

Combining these two with equation 7 we conclude:

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s}.$$

□

References

[1] Tom M Apostol. *Introduction to analytic number theory*. Springer Science & Business Media, 2013.