Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$*.*

Definition 2. *For any positive integer* n*, with prime factorization* $n = p_1^{a_1} p_1^{a_2} \cdots p_k^{a_k}$*, call* $\Omega(n) = a_1 + a_2 + \cdots a_k$ *the number of primes in the factorization counted with multiplicity.*
The ***Liouville Function*** $\lambda(n)$ *is defined as:*

$$\lambda(n) := \begin{cases} 1 & \text{if } n = 1; \\ (-1)^{\Omega(n)} & \text{if } n \neq 1. \end{cases} \tag{2}$$

Theorem 1. *The Riemann Zeta function satisfies the equation:*

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \tag{3}$$

for $Re(s) > 1$*.*

Proof. It is easily seen that the Liouville Function is completely multiplicative, that is to say:

$$\lambda(pq) = \lambda(p)\lambda(q) \quad \text{for all } p, q \in \mathbb{N}.$$

Hence:

$$\lambda(n) = \lambda(p_1^{a_1})\lambda(p_2^{a_2}) \cdots \lambda(p_k^{a_k})$$

and in particular:

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p \sum_{k=0}^{\infty} \frac{\lambda(p^k)}{p^{ks}} \tag{4}$$

this is true because summing over all natural numbers is exactly the same as summing over all possible combinations of products of prime numbers, which is exactly what is happening on the right side of the equation.

Remark 1. *This is identical to the reasoning used in the demonstration of Euler's product formula.*

Therefore, using the definition of the Liouville Function:

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p \sum_{k=0}^{\infty} \frac{\lambda(p^k)}{p^{ks}} = \prod_p \sum_{k=0}^{\infty} \frac{(-1)^k}{p^{ks}}. \tag{5}$$

When $Re(s) > 1$, $\left|\frac{1}{p^s}\right| < 1$, so we have an alternating geometric series converging to $\frac{1}{1+\frac{1}{p^s}}$. Notice also that:

$$\left(1 - \frac{1}{p^s}\right)\left(1 + \frac{1}{p^s}\right) = 1 - \frac{1}{p^{2s}} \Rightarrow \frac{1}{1 + \frac{1}{p^s}} = \frac{1 - \frac{1}{p^s}}{1 - \frac{1}{p^{2s}}}$$

$$\Downarrow$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} &= \prod_p \sum_{k=0}^{\infty} \frac{(-1)^k}{p^{ks}} = \prod_p \frac{1}{1 + \frac{1}{p^s}} = \prod_p \frac{1 - \frac{1}{p^s}}{1 - \frac{1}{p^{2s}}} \\ &= \prod_p \frac{\frac{1}{1-\frac{1}{p^{2s}}}}{\frac{1}{1-\frac{1}{p^s}}} = \frac{\zeta(2s)}{\zeta(s)}. \end{aligned} \tag{6}$$

Where we used the Euler product formula:

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}} \tag{7}$$

true for $Re(s) > 1$.

$\square$