Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1.

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{s(s-1)\Gamma\left(\frac{s}{2}\right)} + \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \int_1^{\infty} \frac{\omega(x)}{x} \left(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right) dx \tag{2}$$

for $s \neq 1$, *where* $\omega(x) := \sum_{n=1}^{\infty} e^{-n^2 \pi x}$.

Proof of this Theorem can be found in [1]. Building on Titchmarsh's demonstration, we have added some meaningful details.

Proof. Begin by observing that if $Re(s) > 0$

$$\int_0^{\infty} x^{\frac{s}{2}-1} e^{-n^2 \pi x} dx = \int_0^{\infty} \left(\frac{y}{n^2 \pi}\right)^{\frac{s}{2}-1} e^{-y} \frac{dy}{n^2 \pi} = \frac{1}{n^s \pi^{\frac{s}{2}}} \int_0^{\infty} y^{\frac{s}{2}-1} e^{-y} dy = \frac{\Gamma\left(\frac{s}{2}\right)}{n^s \pi^{\frac{s}{2}}}$$

where in the last equality we used the Definition of the Gamma function.

Hence if $Re(s) > 1$ we can sum over all $n \in \mathbb{N}$ on both sides to obtain:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}} = \zeta(s) \frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}} = \sum_{n=1}^{\infty} \int_0^{\infty} x^{\frac{s}{2}-1} e^{-n^2 \pi x} dx = \int_0^{\infty} x^{\frac{s}{2}-1} \cdot \left(\sum_{n=1}^{\infty} e^{-n^2 \pi x}\right) dx \tag{3}$$

here the inversion of the order of summation and integration is justified by absolute convergence as, for $Re(s) > 1$:

$$\sum_{n=1}^{\infty} \int_0^{\infty} x^{\frac{Re(s)}{2}-1} e^{-n^2 \pi x} dx = \frac{\Gamma\left(\frac{Re(s)}{2}\right) \zeta(Re(s))}{\pi^{\frac{Re(s)}{2}}}$$

converges.

Define now $\omega(x)$ as:

$$\omega(x) := \sum_{n=1}^{\infty} e^{-n^2\pi x}$$

and write equation 3 as:

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \int_0^{\infty} x^{\frac{s}{2}-1}\omega(x)dx. \tag{4}$$

Consider briefly the Theta function:

$$\theta(x) := \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}.$$

It's clear that $e^{-\pi(-n)^2 x} = e^{-\pi n^2 x}$ and therefore $\sum_{n=0}^{\infty} e^{-\pi n^2 x} = \sum_{n=0}^{-\infty} e^{-\pi n^2 x}$, hence:

$$\begin{aligned} \theta(x) &= \sum_{n=1}^{\infty} e^{-\pi n^2 x} + \sum_{n=0}^{-\infty} e^{-\pi n^2 x} = \omega(x) + \sum_{n=0}^{\infty} e^{-\pi n^2 x} \\ &= \omega(x) + \sum_{n=1}^{\infty} e^{-\pi n^2 x} + 1 = 2\omega(x) + 1. \end{aligned} \tag{5}$$

Notice now that the function $f(s) = e^{-s\pi x^2}$ is a **Schwartz Function** and therefore we can apply to it Poisson's summation formula:

Theorem 2 (Poisson's Summation Formula)**.**
If $f(s)$ is a Schwartz function and $\widehat{f(s)}$ is its Fourier transform, then:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \widehat{f(k)}. \tag{6}$$

In our case $f(n) = e^{-\pi n^2 x}$, with $x > 0$, therefore

$$\widehat{f(n)} = \int_{-\infty}^{\infty} f(x)e^{-2\pi i n x}dx = \frac{1}{\sqrt{x}}e^{-\pi\frac{n^2}{x}}.$$

Hence, applying Theorem 6 to equation 5 implies:

$$\begin{aligned} \theta(x) &= \sum_{n=-\infty}^{\infty} f(n) \\ &= \sum_{n=-\infty}^{\infty} \widehat{f(n)} \\ &= \frac{1}{\sqrt{x}} \sum_{n=-\infty}^{\infty} e^{-\pi\frac{n^2}{x}} = \frac{1}{\sqrt{x}}\theta\left(\frac{1}{x}\right) \\ &= \frac{1}{\sqrt{x}}\left[2\omega\left(\frac{1}{x}\right) + 1\right]. \end{aligned} \tag{7}$$

This, combined with equation 5, proves an important property of $\omega(x)$:

$$2\omega(x) + 1 = \frac{1}{\sqrt{x}}\left[2\omega\left(\frac{1}{x}\right) + 1\right]$$

$$\Downarrow$$

$$\omega(x) = \frac{1}{\sqrt{x}}\omega\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2}.$$

Going back to equation 4 we find:

$$\begin{aligned}
\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} &= \int_0^\infty x^{\frac{s}{2}-1}\omega(x)dx = \int_0^1 x^{\frac{s}{2}-1}\omega(x)dx + \int_1^\infty x^{\frac{s}{2}-1}\omega(x)dx \\
&= \int_0^1 x^{\frac{s}{2}-1}\left[\frac{1}{\sqrt{x}}\omega\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2}\right]dx + \int_1^\infty x^{\frac{s}{2}-1}\omega(x)dx \\
&= \int_0^1 \frac{x^{\frac{s}{2}-1}}{\sqrt{x}}\omega\left(\frac{1}{x}\right)dx + \int_0^1 \frac{x^{\frac{s}{2}-1}}{2\sqrt{x}}dx - \frac{1}{2}\int_0^1 x^{\frac{s}{2}-1}dx + \int_1^\infty x^{\frac{s}{2}-1}\omega(x)dx \\
&= \int_0^1 x^{\frac{s}{2}-1}\frac{1}{\sqrt{x}}\omega\left(\frac{1}{x}\right)dx + \frac{1}{2}\int_0^1 x^{\frac{s}{2}-\frac{3}{2}}dx - \frac{1}{s} + \int_1^\infty x^{\frac{s}{2}-1}\omega(x)dx \\
&= \int_0^1 x^{\frac{s}{2}-1}\frac{1}{\sqrt{x}}\omega\left(\frac{1}{x}\right)dx + \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty x^{\frac{s}{2}-1}\omega(x)dx.
\end{aligned} \tag{8}$$

Changing variable in the first integral to $y = \frac{1}{x}$:

$$\begin{aligned}
\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} &= \frac{1}{s(s-1)} + \int_\infty^1 \left(y^{-1}\right)^{\frac{s}{2}-1}\sqrt{y}\cdot\omega(y)\left(-\frac{dy}{y^2}\right) + \int_1^\infty x^{\frac{s}{2}-1}\omega(x)dx \\
&= \frac{1}{s(s-1)} - \int_1^\infty \frac{y^{-\frac{s}{2}+\frac{3}{2}}}{y^2}\omega(y)\,(-dy) + \int_1^\infty x^{\frac{s}{2}-1}\omega(x)dx \\
&= \frac{1}{s(s-1)} + \int_1^\infty y^{-\frac{s}{2}-\frac{1}{2}}\omega(y)dy + \int_1^\infty x^{\frac{s}{2}-1}\omega(x)dx \\
&= \frac{1}{s(s-1)} + \int_1^\infty \left(x^{-\frac{s}{2}-\frac{1}{2}} + x^{\frac{s}{2}-1}\right)\omega(x)dx \\
&= \frac{1}{s(s-1)} + \int_1^\infty \frac{\omega(x)}{x}\left(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right)dx.
\end{aligned} \tag{9}$$

Which is exactly 2.

The last integral converges for all values of s and so the formula holds, by analytic continuation, for all $s \neq 1$. □

Notice that, computing the right-hand side for $1-s$ gives us:

$$\begin{aligned}
&\frac{1}{(1-s)(1-s-1)} + \int_1^\infty \left(x^{-\frac{(1-s)}{2}-\frac{1}{2}} + x^{\frac{(1-s)}{2}-1}\right)\omega(x)dx \\
&= \frac{1}{s(1-s)} + \int_1^\infty \left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}\right)\omega(x)dx
\end{aligned} \tag{10}$$

So the right-hand side of 8 is unchanged if s is replaced by $1 - s$, therefore so is the left-hand side which means that:

$$\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} = \zeta(1-s)\Gamma\left(\frac{1-s}{2}\right)\pi^{\frac{s-1}{2}}$$

this is a less common form of Riemann's Functional Equation.

References

[1] Edward Charles Titchmarsh and David Rodney Heath-Brown. *The theory of the Riemann zeta-function*. Oxford university press, 1986.